

SYMMETRIC CONVEX SETS WITH MINIMAL GAUSSIAN SURFACE AREA

ABSTRACT. Let $\Omega \subseteq \mathbb{R}^{n+1}$ have minimal Gaussian surface area among all sets satisfying $\Omega = -\Omega$ with fixed Gaussian volume. Let $A = A_x$ be the second fundamental form of $\partial\Omega$ at x , i.e. A is the matrix of first order partial derivatives of the unit normal vector at $x \in \partial\Omega$. For any $x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$, let $\gamma_n(x) = (2\pi)^{-n/2} e^{-(x_1^2 + \dots + x_{n+1}^2)/2}$. Let $\|A\|^2$ be the sum of the squares of the entries of A , and let $\|A\|_{2 \rightarrow 2}$ denote the ℓ_2 operator norm of A .

It is shown that if Ω or Ω^c is convex, and if either

$$\int_{\partial\Omega} (\|A_x\|^2 - 1) \gamma_n(x) dx > 0 \quad \text{or} \quad \int_{\partial\Omega} \left(\|A_x\|^2 - 1 + 2 \sup_{y \in \partial\Omega} \|A_y\|_{2 \rightarrow 2}^2 \right) \gamma_n(x) dx < 0,$$

then $\partial\Omega$ must be a round cylinder. That is, except for the case that the average value of $\|A\|^2$ is slightly less than 1, we resolve the convex case of a question of Barthe from 2001.

The main tool is the Colding-Minicozzi theory for Gaussian minimal surfaces, which studies eigenfunctions of the Ornstein-Uhlenbeck type operator $L = \Delta - \langle x, \nabla \rangle + \|A\|^2 + 1$ associated to the surface $\partial\Omega$. A key new ingredient is the use of a randomly chosen degree 2 polynomial in the second variation formula for the Gaussian surface area. Our actual results are a bit more general than the above statement. Also, some of our results hold without the assumption of convexity.

1. INTRODUCTION

Below, $\Sigma \subseteq \mathbb{R}^{n+1}$ will always denote an n -dimensional orientable C^∞ hypersurface with outward pointing unit normal vector N . Define the **mean curvature** at $x \in \Sigma$ to be $H(x) := \operatorname{div}(N(x))$.

Example 1.1. Define $S^n \subseteq \mathbb{R}^{n+1}$ so that

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Let $r > 0$. If $\Sigma := rS^n$, then $H(x) = n/r$ for all $x \in \Sigma$.

Let $\{\Sigma_s\}_{s \in (-1,1)}$ be a set of hypersurfaces such that $\Sigma_0 = \Sigma$. Then the first variation formula for Euclidean surface area says

$$\frac{d}{ds} \Big|_{s=0} \int_{\Sigma_s} dx = \int_{\Sigma} \left\langle \frac{\partial}{\partial s} \Big|_{s=0} x, H(x) N(x) \right\rangle dx.$$

So, if we define a sequence of hypersurfaces $\{\Sigma_s\}_{s \geq 0}$ such that

$$\frac{d}{ds} x = -H(x) N(x), \quad \forall x \in \Sigma_s, \quad \forall s \geq 0, \tag{1}$$

then this flow decreases Euclidean volume in the fastest way possible. A family of surfaces $\{\Sigma_s\}_{s \geq 0}$ satisfying (1) is called a **mean curvature flow**.

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In a landmark investigation of mean curvature flow [CM12], Colding and Minicozzi studied the following quantity, which turns out to monotonically decrease under mean curvature flow.

$$\sup_{a>0, b \in \mathbb{R}^{n+1}} \int_{\Sigma} a^{-\frac{n}{2}} \gamma_n((x-b)a^{-1/2}) dx \quad (2)$$

Here, with $m = n + 1$, we define

$$\gamma_n(x) := (2\pi)^{-n/2} e^{-\|x\|^2/2}, \quad \|x\|^2 := \sum_{i=1}^m x_i^2, \quad \forall x = (x_1, \dots, x_m) \in \mathbb{R}^m.$$

$$\int_{\Sigma} \gamma_n(x) dx := \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \int_{\{x \in \mathbb{R}^{n+1} : \exists y \in \Sigma, \|x-y\| < \varepsilon\}} \gamma_n(x) dx.$$

In [CM12], (2) is called the “entropy” of Σ . Note that (2) is a maximal version of the Gaussian perimeter. In the context of mean curvature flow, the Colding-Minicozzi entropy (1) is an analogue of Perelman’s reduced volume for Ricci flow.

Colding-Minicozzi were interested in understanding the singularities that arise in mean curvature flow. Since (2) monotonically decreases under this flow, it is then sensible to study minimizers of (2). It was conjectured in [CM12] and ultimately proven in [Zhu16] that:

Theorem 1.2 ([Zhu16]). *Among all compact n -dimensional hypersurfaces $\Sigma \subseteq \mathbb{R}^{n+1}$ with $\partial\Sigma = \emptyset$, the round sphere minimizes the quantity (1).*

Definition 1.3. We say that $\{\Sigma_s\}_{s \in (-1,1)}$ is a **normal variation** of Σ if \exists a C^∞ function $f: \Sigma \rightarrow \mathbb{R}$ such that

$$\Sigma_s := \{x + sN(x)f(x) : x \in \Sigma\}, \quad \forall s \in (-1, 1). \quad (3)$$

2. SELF-SHRINKERS

For non-compact minimizers of (2), Colding-Minicozzi restricted their attention to self-shrinkers, which model singularities of mean curvature flow.

Definition 2.1. We say that a hypersurface Σ is a **self-shrinker** if

$$H(x) = \langle x, N(x) \rangle, \quad \forall x \in \Sigma. \quad (4)$$

Example 2.2. Examples of self-shrinkers include a hyperplane through the origin, the sphere $\sqrt{n}S^n$, or more generally, round cylinders $\sqrt{k}S^k \times S^{n-k}$, where $0 \leq k \leq n$, and also cones with zero mean curvature.

It is shown in [CM12, Proposition 3.6] that Σ is a self-shrinker if and only if it is a critical point of Gaussian surface area, in the following sense: for any differentiable $a: (-1, 1) \rightarrow \mathbb{R}$ with $a(0) = 1$, for any differentiable $b: (-1, 1) \rightarrow \mathbb{R}^{n+1}$ with $b(0) = 0$, and for any normal variation $\{\Sigma_s\}_{s \in (-1,1)}$ of Σ (as in (3)), we have

$$\frac{\partial}{\partial s} \Big|_{s=0} \int_{\Sigma_s} (a(s))^{-\frac{n}{2}} \gamma_n((x-b(s))(a(s))^{-1/2}) dx = 0.$$

That is, self-shrinkers are critical points of Gaussian surface area, if we mod out by translations and dilations.

Theorem 2.3 ([CM12, Theorem 0.12]). *Let Σ be a geodesically complete self-shrinker without boundary and such that $\exists c > 0$ such that, $\forall r > 0$, $\int_{\{x \in \Sigma: \|x\| \leq r\}} dx \leq cr^n$. Suppose that, for all $0 \leq k \leq n$, Σ is not isometric to a dilation of $S^k \times \mathbb{R}^{n-k}$. Then there is a graph $\bar{\Sigma}$ over Σ of a function with arbitrarily small C^m norm (for any fixed m) such that the entropy of $\bar{\Sigma}$ is less than that of Σ .*

3. BARTHE'S QUESTION

It is well-known that the set $\Omega \subseteq \mathbb{R}^{n+1}$ of fixed Gaussian volume $\int_{\Omega} \gamma_{n+1}(x) dx$ and of minimal Gaussian surface area $\int_{\partial\Omega} \gamma_n(x) dx$ is a half space. That is, Ω is the set lying on one side of a hyperplane [SC74]. This result has been elucidated and strengthened over the years [Bor85, Led94, Led96, Bob97, BS01, Bor03, MN⁺15b, MN15a, Eld15, MR15, BBJ16].

Barthe [Bar01] asked this same question with the addition restriction that Ω is **symmetric**, i.e. that $\Omega = -\Omega$.

Problem 3.1. *Fix $0 < c < 1$. Minimize*

$$\int_{\partial\Omega} \gamma_n(x) dx$$

over all subsets $\Omega \subseteq \mathbb{R}^{n+1}$ satisfying $\Omega = -\Omega$ and $\gamma_{n+1}(\Omega) = c$.

Remark 3.2. If Ω minimizes Problem 3.1, then Ω^c also minimizes Problem 3.1

Conjecture 3.3 ([Bar01, CR11, O'D12]). *Suppose $\Omega \subseteq \mathbb{R}^{n+1}$ minimizes Problem 3.1. Then, after rotating Ω , $\exists r > 0$ and $\exists 0 \leq k \leq n$ such that*

$$\partial\Omega = rS^k \times \mathbb{R}^{n-k}.$$

Essentially all known proof methods (with the exception of [MR15, BBJ16]) seem unable to handle the additional constraint that the set Ω is symmetric in Problem 3.1. That is, new methods are needed to find symmetric sets $\Omega \subseteq \mathbb{R}^{n+1}$ of fixed Gaussian volume and minimal Gaussian surface area.

It is well-known that $\exists \lambda \in \mathbb{R}$ such that

$$H(x) = \langle x, N(x) \rangle + \lambda, \quad \forall x \in \Sigma \tag{5}$$

if and only if, for any normal variation of Σ with $\int_{\Sigma} f \gamma_n(x) dx = 0$,

$$\left. \frac{\partial}{\partial s} \right|_{s=0} \int_{\Sigma_s} \gamma_n(x) dx = 0.$$

That is, Σ is a critical point of the Gaussian surface area if and only if (5) holds. Note that (5) is a generalization of (4).

4. SECOND VARIATION

Definition 4.1. Let e_1, \dots, e_n be an orthonormal frame of $\Sigma \subseteq \mathbb{R}^{n+1}$. That is, for a fixed $x \in \Sigma$, there exists a neighborhood U of x such that e_1, \dots, e_n is an orthonormal basis for the tangent space of Σ , for every point in U [Lee03, Proposition 11.17].

Define the **mean curvature**

$$H = H(x) := \operatorname{div}(N(x)) = \sum_{i=1}^n \langle \nabla_{e_i} N(x), e_i \rangle. \tag{6}$$

Define the **second fundamental form** $A = A_x = (a_{ij})_{1 \leq i, j \leq n}$ so that

$$a_{ij} = a_{ij}(x) = \langle \nabla_{e_i} e_j, N(x) \rangle, \quad \forall 1 \leq i, j \leq n. \quad (7)$$

A key aspect of the Colding-Minicozzi theory is the study of the second variation of the Gaussian perimeter. More specifically, we study of eigenfunctions of the differential operator L , defined for any C^∞ function $f: \Sigma \rightarrow \mathbb{R}$ by

$$Lf(x) := \Delta f(x) - \langle x, \nabla f(x) \rangle + \|A_x\|^2 f(x) + f(x), \quad \forall x \in \Sigma. \quad (8)$$

Here $\|A_x\|^2$ is the sum of the squares of the entries of the matrix A_x . Note that L is an Ornstein-Uhlenbeck-type operator.

Example 4.2. If Σ is a hyperplane, then $A_x = 0$ for all $x \in \Omega$, so L is exactly the usual Ornstein-Uhlenbeck operator, plus the identity map. So, the eigenfunctions of L are the Hermite polynomials.

If $r > 0$ and if $\Sigma = rS^n$, then $\|A_x\|^2 = n/r^2$. And the eigenfunctions of L are spherical harmonics. This is most easily seen by integrating by parts:

$$\int_{\Sigma} f Lf \gamma_n(x) dx = \int_{\Sigma} (-\|\nabla f\|^2 + f^2(\|A\|^2 + 1)) \gamma_n(x) dx.$$

The work [CM12] made the following crucial observation about the operator L . If (4) holds, then H is an eigenfunction of L with eigenvalue 2:

$$(4) \implies LH = 2H. \quad (9)$$

The Colding-Minicozzi theory can readily solve Problem 3.1 in the special case that (4) holds (which is more restrictive than (5)). For illustrative purposes, we now sketch this argument, which closely follows [CM12, Theorem 4.30]. In particular, we use the following key insights of [CM12].

- H is an eigenfunction of L with eigenvalue 2. (That is, (9) holds.)
- The second variation formula for Gaussian surface area is a quadratic form involving L .
- If H changes sign, then an eigenfunction of L exists with eigenvalue larger than 2.

Proposition 4.3 (Special Case of Conjecture 3.3). *Let $\Omega \subseteq \mathbb{R}^{n+1}$ minimize Problem 3.1. As noted in (5), $\exists \lambda \in \mathbb{R}$ such that (5) holds. Assume $\lambda = 0$. Assume also that $\Sigma := \partial\Omega$ is a compact, C^∞ hypersurface. Then $\exists r > 0$ such that $\partial\Omega = rS^n$.*

Proof. Let H be the mean curvature of Σ . If $H \geq 0$, then Huisken's classification [Hui90, Hui93] [CM12, Theorem 0.17] of compact surfaces satisfying (4) implies that Σ is a round sphere ($\exists r > 0$ such that $\Sigma = rS^n$). So, we may assume that H changes sign. As noted in (9), $LH = 2H$. Since H changes sign, 2 is not the largest eigenvalue of L , by spectral theory [Zhu16, Lemma 6.5] (e.g. using that $(L - \|A\|^2 - 2)^{-1}$ is a compact operator). That is, there exists a C^2 function $g: \Sigma \rightarrow \mathbb{R}$ and there exists $\delta > 2$ such that $Lg = \delta g$. Moreover, $g > 0$ on Σ . Since $g > 0$ and $\Sigma = -\Sigma$, it follows by (8) that $g(x) + g(-x)$ is an eigenfunction of L with eigenvalue δ . That is, we may assume that $g(x) = g(-x)$ for all $x \in \Sigma$.

Since Σ is not a round sphere, it suffices to find a nearby hypersurface of smaller Gaussian surface area. For any C^2 function $f: \Sigma \rightarrow \mathbb{R}$, and for any $s \in (-1, 1)$, consider the hypersurface

$$\Sigma_s := \{x + sN(x)f(x) : x \in \Sigma\}. \quad (10)$$

From the second variation formula for Gaussian surface area

$$\frac{d^2}{ds^2}\Big|_{s=0} \int_{\Sigma_s} \gamma_n(x) dx = - \int_{\Sigma} f(x) Lf(x) \gamma_n(x) dx.$$

So, to complete the proof, it suffices by the second variation formula to find a C^2 function f such that

- $f(x) = f(-x)$ for all $x \in \Sigma$. (f preserves symmetry.)
- $\int_{\Sigma} f(x) \gamma_n(x) dx = 0$. (f preserves Gaussian volume.)
- $\int_{\Sigma} f(x) Lf(x) \gamma_n(x) dx > 0$. (f decreases Gaussian surface area.)

We choose g as above so that $Lg = \delta g$, $\delta > 2$ and so that $\int_{\Sigma} (H(x) + g(x)) \gamma_n(x) dx = 0$. (Since H changes sign and $g \geq 0$, g can satisfy the last equality by multiplying it by an appropriate constant.) We then define $f := H + g$. Then f satisfies the first two properties. So, it remains to show that f satisfies the last property. Note that, since H and g have different eigenvalues, they are orthogonal, i.e. $\int_{\Sigma} H(x) g(x) \gamma_n(x) dx = 0$. Therefore,

$$\begin{aligned} \int_{\Sigma} f(x) Lf(x) \gamma_n(x) dx &\stackrel{(9)}{=} \int_{\Sigma} (H(x) + g(x)) (2H(x) + \delta g(x)) \gamma_n(x) dx \\ &= 2 \int_{\Sigma} (H(x))^2 \gamma_n(x) dx + \delta \int_{\Sigma} (g(x))^2 \gamma_n(x) dx > 0. \end{aligned}$$

(Since $H(x) = \langle x, N(x) \rangle$ for all $x \in \Sigma$, and Σ is compact, both $\int_{\Sigma} (H(x))^2 \gamma_n(x) dx$ and $\int_{\Sigma} H(x) \gamma_n(x) dx$ exist.) \square

Remark 4.4. The case that $\partial\Omega$ is not compact can also be dealt with [CM12, Lemmas 9.44 and 9.45], [Zhu16, Proposition 6.11]. Instead of asserting the existence of g , one approximates g by a sequence of Dirichlet eigenfunctions on the intersection of Σ with large compact balls. Since Proposition 4.3 was presented only for illustrative purposes, and since the assumption (4) is too restrictive to resolve Problem 3.1, we will not present the details.

Unfortunately, the proof of Proposition 4.3 does not extend to the more general assumption (5). In order to attack Problem 3.1, we can only assume that (5) holds, instead of the more restrictive (4).

Under the assumption of (5), the proof of Proposition 4.3 breaks in at least two significant ways. First, H is no longer an eigenfunction of L when (5) holds with $\lambda \neq 0$.

Second, Huisken's classification no longer holds [Hui90, Hui93]. Indeed, it is known that, for every integer $m \geq 3$, there exists $\lambda = \lambda_m < 0$ and there exists a convex embedded curve $\Gamma_m \subseteq \mathbb{R}^2$ satisfying (5) and such that Γ_m has m -fold symmetry (and $\Gamma_{m_1} \neq \Gamma_{m_2}$ if $m_1 \neq m_2$) [Cha17, Theorem 1.3, Proposition 3.2]. Consequently, $\Gamma_m \times \mathbb{R}^{n-2} \subseteq \mathbb{R}^{n+1}$ also satisfies (5). That is, Huisken's classification cannot possibly hold, at least when $\lambda < 0$ in (5).

5. OUR CONTRIBUTION

Our first result shows that Huisken's classification does actually hold for surfaces satisfying (5) if $\lambda > 0$, if the surface Σ encloses a convex region.

Theorem 5.1 (Huisken-type classification, $\lambda > 0$). *Let $\Omega \subseteq \mathbb{R}^{n+1}$ minimize Problem 3.1 and let $\Sigma := \partial\Omega$. From (5), $H(x) = \langle x, N(x) \rangle + \lambda$ for all $x \in \partial\Omega$. Assume $\lambda > 0$. Then, after rotating Ω , $\exists r > 0$ and $\exists 0 \leq k \leq n$ such that $\partial\Omega = rS^k \times \mathbb{R}^{n-k}$.*

Related to Huisken's classification [Hui90, Hui93] [CM12, Theorem 0.17] are Bernstein theorems. If a hypersurface Σ satisfies (4) and Σ can be written as the graph of a function, then Σ is a hyperplane [EH89] [Wan11]. Also, if a hypersurface Σ satisfies (5), if Σ has polynomial volume growth and if Σ be written as the graph of a function, then Σ is a hyperplane [Gua, Theorem 1.6] [CW14, Theorem 1.3]. In particular, if $\Omega = -\Omega$, if $\partial\Omega$ satisfies (5), and if Ω can be separated by a hyperplane into two sets, each of which is the graph of a function, then $\partial\Omega$ must consists of two parallel hyperplanes. In this sense, the symmetric strip separated by two parallel hyperplanes (or its complement) are "isolated critical points" in Problem 3.1.

Due to the Bernstein-type theorems of [Gua, CW14], the main difficulty of Problem 3.1 occurs when Σ is not the graph of a function. Also, by Theorem 5.1 and (5), in order to solve the convex case of Problem 3.1, it suffices to restrict to surfaces Σ such that there exists $\lambda < 0$ and such that $H(x) = \langle x, N(x) \rangle + \lambda$, for all $x \in \Sigma$. As discussed above, the case $\lambda < 0$ is most interesting, since a Huisken-type classification cannot possibly hold when $\lambda < 0$. To deal with the case $\lambda < 0$, we use second variation arguments, as in [CM12, CIMW13].

We begin by using the mean curvature minus its mean in the second variation formula for Gaussian surface area.

Theorem 5.2 (Second Variation Using an Eigenfunction of L , $\lambda < 0$). *Let Ω minimize Problem 3.1 and let $\Sigma := \partial\Omega$. Then by the first variation formula (5), $\exists \lambda \in \mathbb{R}$ such that $H(x) = \langle x, N(x) \rangle + \lambda$, for any $x \in \Sigma$. Assume $\lambda < 0$. If*

$$\int_{\Sigma} (\|A_x\|^2 - 1)\gamma_n(x)dx > 0,$$

then, after rotating Ω , $\exists r > 0$ and $0 \leq k \leq n$ so that $\Sigma = rS^k\mathbb{R}^{n-k}$.

To handle the case when the average curvature of Σ is less than 1, we use our intuition about the sphere itself. On the sphere, the mean zero symmetric eigenfunctions of L which maximize the second variation of Gaussian surface area are degree two homogeneous spherical harmonics. This was observed in [Man17]. A similar observation was made in the context of noise stability in [Hei15]. In fact, if $v, w \in S^n$, if $\langle v, w \rangle = 0$ and if $\Sigma = S^n$ then $\langle v, N \rangle \langle w, N \rangle$ is an eigenfunction of L . So, intuitively, if $\exists \lambda \in \mathbb{R}$ such that $H(x) = \langle x, N(x) \rangle + \lambda$, then $\langle v, N \rangle \langle w, N \rangle$ should also be an eigenfunction of L . Unfortunately, this does not seem to be true. Nevertheless, if we average over all possible choices of $v, w \in S^n$, then we can obtain a good bound in the second variation formula. And then there must exist $v, w \in S^n$ whose second variation exceeds this average value.

If $y \in \Sigma$, then we define $\|A_y\|_{2 \rightarrow 2}^2 := \sup_{v \in S^n} \|A_y v\|^2$ to be the ℓ_2 operator norm of A_y . Also, $\Pi_y: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ denotes the linear projection onto the tangent space of $y \in \Sigma$. (So $\|\Pi_y v\|^2 = 1 - \langle N(y), v/\|v\| \rangle$ for any $v \in \mathbb{R}^{n+1} \setminus \{0\}$.)

As noted in Proposition 4.3, Problem 3.1 reduces to finding functions $f: \Sigma \rightarrow \mathbb{R}$ such that $\int_{\Sigma} f L f \gamma_n(x) dx$ is as large as possible.

Theorem 5.3 (Second Variation Using a Random Bilinear Function). *Let $\Sigma \subseteq \mathbb{R}^{n+1}$ be an orientable hypersurface with $\partial\Sigma = \emptyset$. Suppose $\exists \lambda \in \mathbb{R}$ such that $H(x) = \langle x, N(x) \rangle + \lambda$ for all $x \in \Sigma$. Let $p := \int_{\Sigma} \gamma_n(x) dx$.*

There exists $v, w \in S^n$ so that, if $m := \frac{1}{p} \int_{\Sigma} \langle v, N \rangle \langle w, N \rangle \gamma_n(y) dy$, we have

$$\begin{aligned} & (n+1)^2 \int_{\Sigma} (\langle v, N \rangle \langle w, N \rangle - m) L(\langle v, N \rangle \langle w, N \rangle - m) \gamma_n(x) dx \\ & \geq \frac{1}{p^2} \int_{\Sigma \times \Sigma \times \Sigma} (1 - \|A_x\|^2 - 2 \|A_y\|_{2 \rightarrow 2}^2) \|\Pi_y(N(z))\|^2 \gamma_n(x) \gamma_n(y) \gamma_n(z) dx dy dz. \end{aligned}$$

Note that convexity is not assumed in Theorem 5.3.

Theorem 5.3 actually follows from a slightly more general statement.

Theorem 5.3 is sharp for spheres, as observed by [Man17]. If $r > 0$, and if $\Sigma = rS^n$, then $\|A_x\|^2 = n/r^2$ and $\|A_y\|_{2 \rightarrow 2}^2 = 1/r^2$, so

$$1 - \|A_x\|^2 - 2 \|A_y\|_{2 \rightarrow 2}^2 = \frac{r^2 - n - 2}{r^2}.$$

If $v, w \in S^n$ satisfy $\langle v, w \rangle = 0$, then $m = 0$ and $\int_{\Sigma} \langle v, N \rangle \langle w, N \rangle L(\langle v, N \rangle \langle w, N \rangle) \gamma_n(x) dx \geq 0$ if and only if $r \geq \sqrt{n+2}$ [Man17, Proposition 1].

Since Theorem 5.3 gives a bound on the second variation, Theorem 5.3 implies the following.

Corollary 5.4 (Second Variation Using a Random Bilinear Function). *Let Ω minimize Problem 3.1 and let $\Sigma := \partial\Omega$, so that $\exists \lambda \in \mathbb{R}$ such that $H(x) = \langle x, N(x) \rangle + \lambda$ for all $x \in \Sigma$. If*

$$\int_{\Sigma \times \Sigma \times \Sigma} (1 - \|A_x\|^2 - 2 \|A_y\|_{2 \rightarrow 2}^2) \|\Pi_y(N(z))\|^2 \gamma_n(x) \gamma_n(y) \gamma_n(z) dx dy dz > 0,$$

then, after rotating Ω , $\exists r > 0$ and $\exists 0 \leq k \leq n$ so that $\Sigma = rS^k \times \mathbb{R}^{n-k}$.

The combination of Remark 3.2, Theorems 5.1 and 5.2 and Corollary 5.4 implies the following.

Theorem 5.5 (Main Result). *Let Ω minimize Problem 3.1 and let $\Sigma := \partial\Omega$. Assume that Ω or Ω^c is convex. If*

$$\int_{\Sigma} (\|A_x\|^2 - 1) \gamma_n(x) dx > 0,$$

or

$$\int_{\Sigma} (\|A_x\|^2 - 1 + 2 \sup_{y \in \Sigma} \|A_y\|_{2 \rightarrow 2}^2) \gamma_n(x) dx < 0,$$

then, after rotating Ω , $\exists r > 0$ and $\exists 0 \leq k \leq n$ so that $\Sigma = rS^k \times \mathbb{R}^{n-k}$.

So, except for the case that the average value of $\|A\|^2$ is slightly less than 1, we resolve the convex case of Barthe's Conjecture 3.3.

We also adapt an argument of [CM12] that allows the computation of the second variation of Gaussian volume preserving normal variations, which simultaneously can dilate the hypersurface Σ . When we use the function $H - \lambda$ in this second variation formula, we get zero. This suggests the intriguing possibility that the fourth variation of $H - \lambda$ could help to solve Problem 3.1. Instead of embarking on a rather technical enterprise of computing Gaussian volume preserving fourth variations, we instead put the function $H - \lambda + t$, $t \in \mathbb{R}$ into this second variation formula, and we then differentiate twice in t . We then arrive at the following interesting inequality.

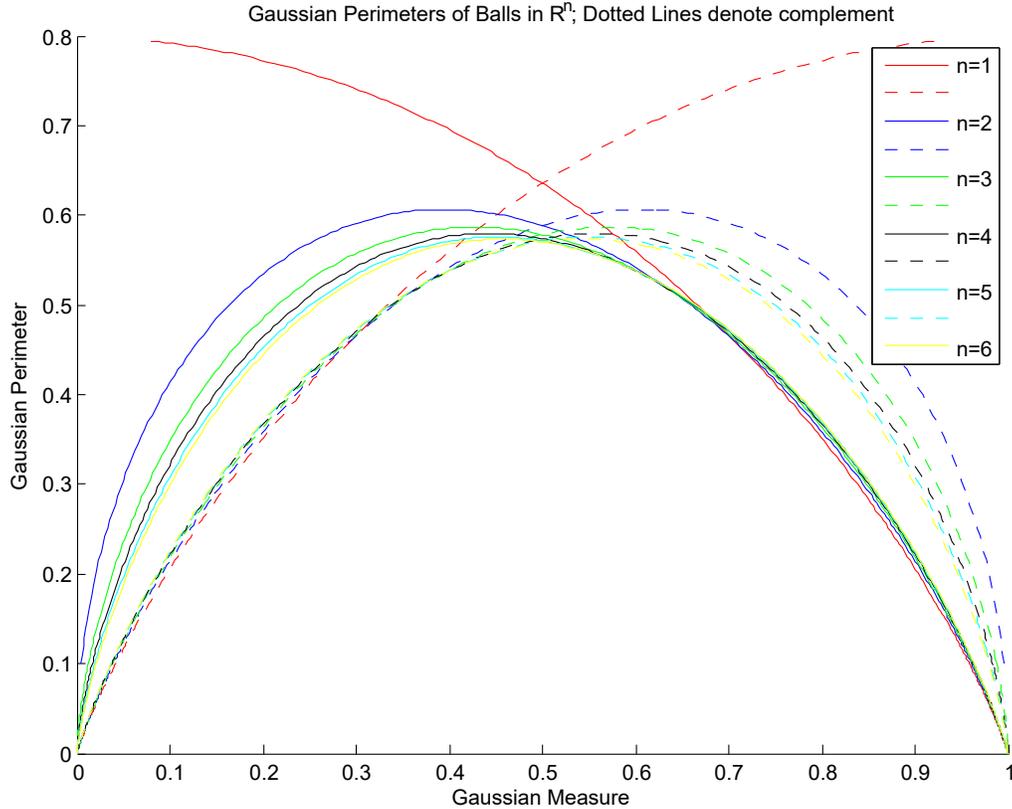
Theorem 5.6. *Let Ω minimize Problem 3.1 and let $\Sigma := \partial\Omega$. Assume also that Σ is a compact, C^∞ hypersurface and Ω is convex. Then*

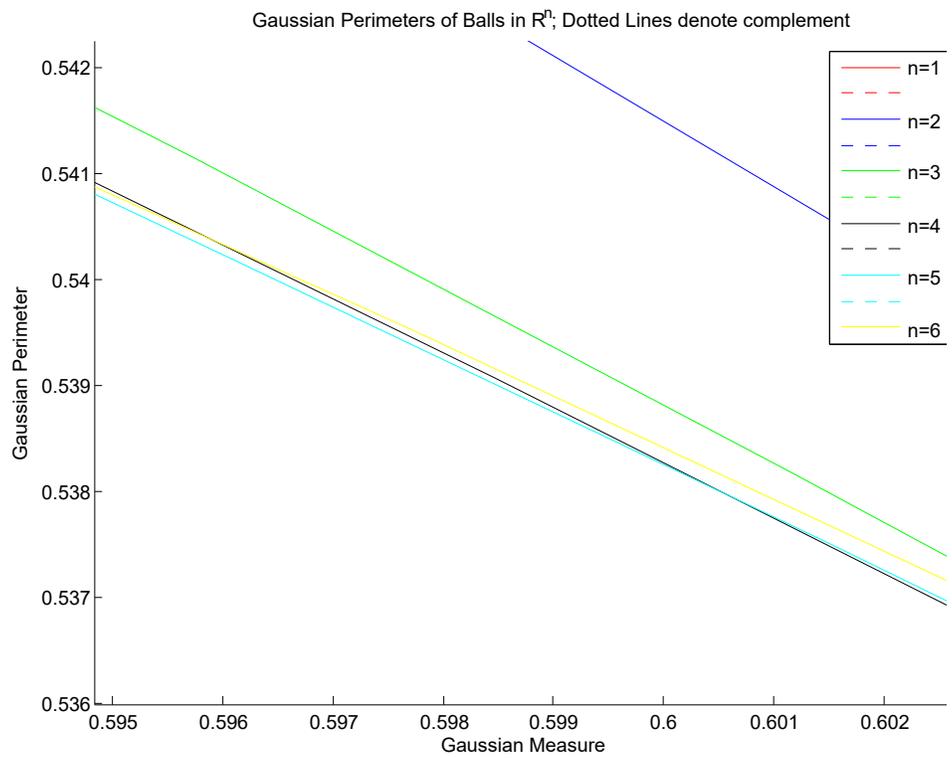
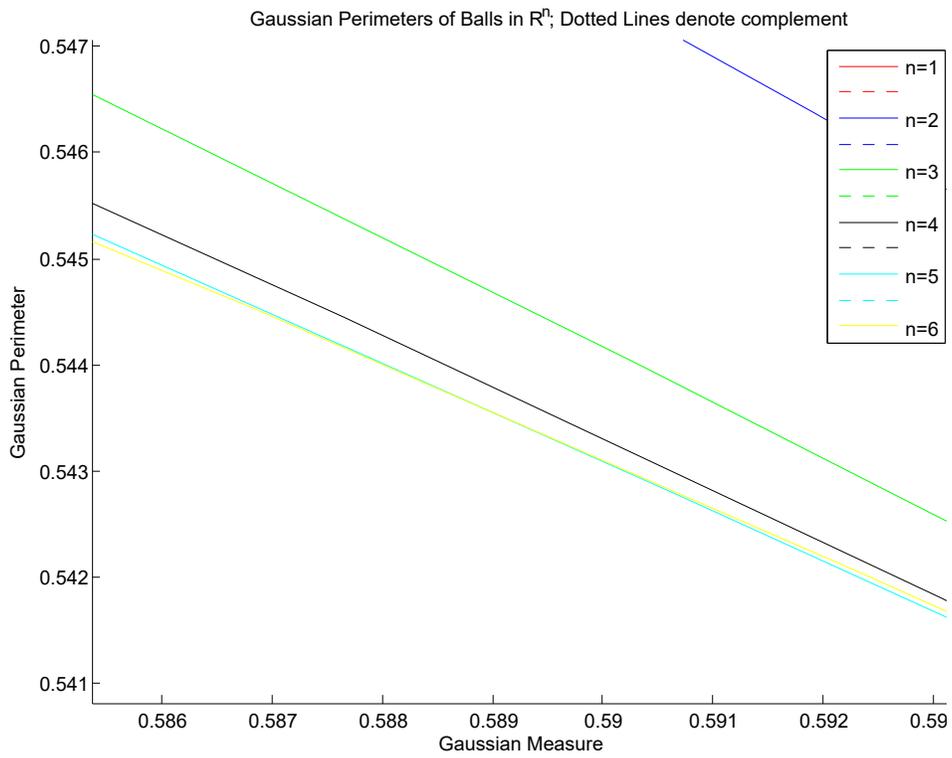
$$\int_{\Sigma} \left(-\|A_x\|^2 + H(x) \frac{\int_{\Sigma} \gamma_n(z) dz}{\int_{\Sigma} \langle y, N \rangle \gamma_n(y) dy} \right) \gamma_n(x) dx \geq 0.$$

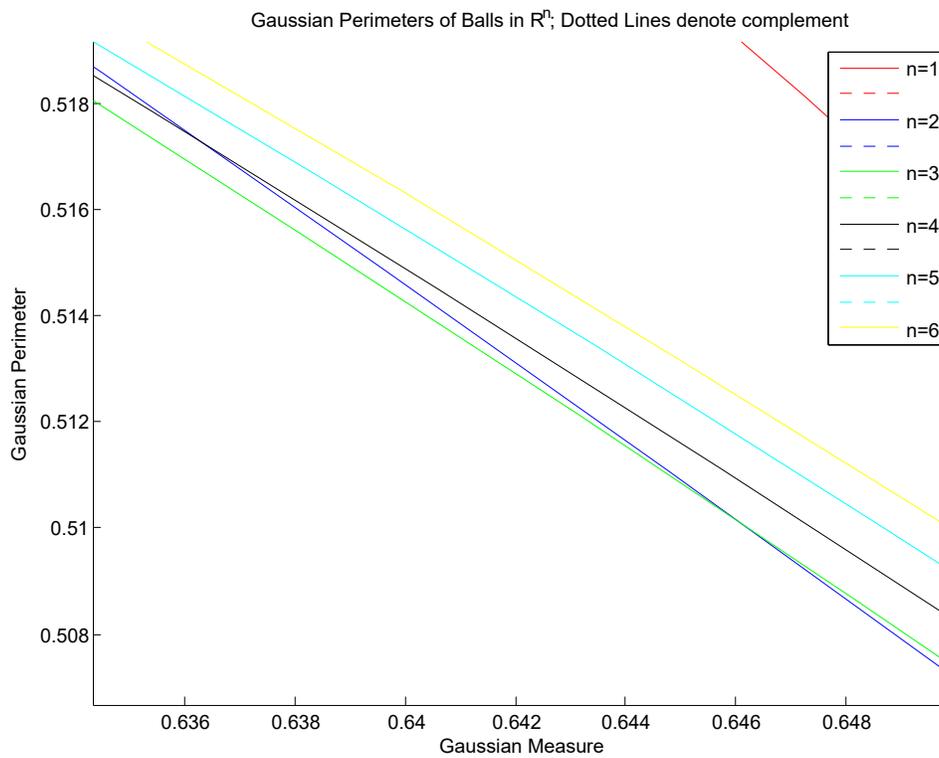
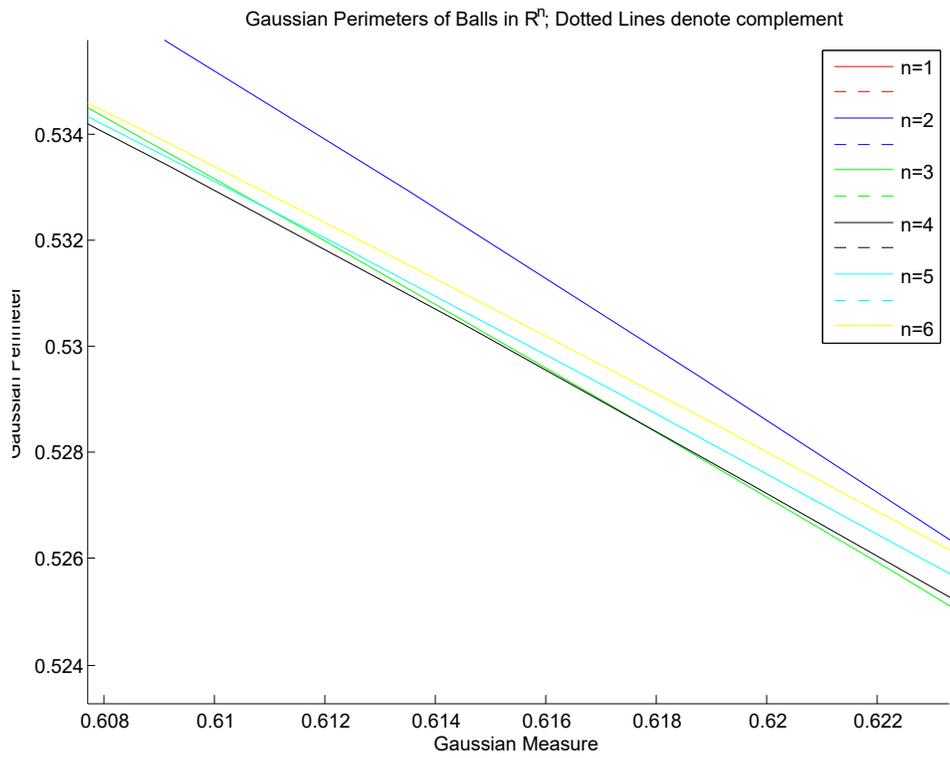
This inequality is rather interesting since it is equal to zero exactly when $\Sigma = rS^n$, for any $r > 0$, since then $\|A_x\|^2 = n/r^2$, $H(x) = n/r$, and $\langle x, N \rangle = r$ for all $x \in S^n$. So, one might speculate that round spheres are the only compact C^∞ hypersurfaces, where this quantity is nonnegative, and where $\exists \lambda \in \mathbb{R}$ such that $H(x) = \langle x, N(x) \rangle + \lambda$ for all $x \in \Sigma$.

6. NUMERICAL COMPUTATIONS

The following numerical computations plot the Gaussian perimeter $(d/dr) \int_{B(0,r)} \gamma_n(x) dx$ versus the Gaussian volume $\int_{B(0,r)} \gamma_n(x) dx$ for $n = 1, \dots, 6$, where $B(0, r) = \{x \in \mathbb{R}^n : \|x\| \leq r\}$. These computations seem to suggest that there exists a sequence of intervals $\dots [a_3, a_2] \cup [a_2, a_1] \cup [a_1, 1]$ with $a_0 = 1$ such that $[1/2, 1] \supseteq \cup_{n=1}^\infty [a_n, a_{n-1}]$ and such that, for every $n \geq 1$, and for every $p \in [a_n, a_{n-1}]$, the minimum Gaussian perimeter of all balls of Gaussian measure p occurs for the ball in \mathbb{R}^n . (Thanks to Frank Morgan for suggesting this possibility.)







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%%%%
% cyltest.m  plot the gaussian perimeters of cylinders of
% different dimensions and of different gaussian measures
%
% the Gaussian perimeter of a ball in R^n of radius r is
%      r^(n-1) n / [2^(n/2) Gamma(1+ n/2)]
% the Gaussian volume of a ball in R^n of radius r is
%      t_(n-1)(r) n / [2*(n/2) Gamma(1 + n/2)]
% where t_(0)(r)=sqrt(pi/2)erf(r/sqrt(2))
%      t_(1)(r)= 1- exp(-r^2 /2)
% and t_(n-1)(r) = -r^(n-2) exp(-r^2 /2)+(n-2)t_(n-3)(r)
%
% erf(x) = 2/sqrt(pi) \int_0^x exp(-t^2)dt
% erf(x/sqrt(2))=2/sqrt(pi) \int_0^{x/sqrt(2)} exp(-t^2)dt.  s=sqrt(2)t
%      =sqrt(2/pi) \int_0^x exp(-t^2 /2)dt

r=linspace(.1,100,10000);
close all;
figure;
hold on;

%plot the perimeter vs the measure

n1perim=2*(1/sqrt(2*pi))*exp(-(r.^2)/2);
n1meas=2*(1/2)*erf(r/sqrt(2));
plot(n1meas,n1perim,'r',1-n1meas,n1perim,'r--');

n2perim=r.*exp(-(r.^2)/2);
n2meas=1-exp(-(r.^2)/2);
plot(n2meas,n2perim,'b',1-n2meas,n2perim,'b--');

n3perim=(sqrt(2/pi))*(r.^2).*exp(-(r.^2)/2);
n3meas=(sqrt(2/pi))*(-r.*exp(-(r.^2)/2) + (sqrt(pi/2))*erf(r/sqrt(2)));
plot(n3meas,n3perim,'g',1-n3meas,n3perim,'g--');

n4perim=(1/2)*(r.^3).*exp(-(r.^2)/2);
n4meas=1-(1+((r.^2)/2)).*(exp(-(r.^2)/2));
plot(n4meas,n4perim,'k',1-n4meas,n4perim,'k--');

n5perim=(1/3)*(sqrt(2/pi))*(r.^4).*exp(-(r.^2)/2);
n5meas=(1/3)*(sqrt(2/pi))*(-(r.^3).*(exp(-(r.^2)/2)) +...
+3*(-r.*(exp(-(r.^2)/2))+sqrt(pi/2)*erf(r/sqrt(2))));
plot(n5meas,n5perim,'c',1-n5meas,n5perim,'c--');

n6perim=(1/8)*(r.^5).*exp(-(r.^2)/2);

```

```

n6meas=(1/8)*(-(r.^4).*(exp(-(r.^2)/2)) +4*(-(2+(r.^2)).*(exp(-(r.^2)/2))+2));
plot(n6meas,n6perim,'y',1-n6meas,n6perim,'y--');

legend('n=1',' ','n=2',' ','n=3',' ','n=4',' ','n=5',' ','n=6');
xlabel('Gaussian Measure');
ylabel('Gaussian Perimeter');
title('Gaussian Perimeters of Balls in R^n; Dotted Lines denote complement');

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REFERENCES

- [Bar01] Franck Barthe, *An isoperimetric result for the Gaussian measure and unconditional sets*, Bulletin of the London Mathematical Society **33** (2001), 408–416.
- [BBJ16] Marco Barchiesi, Alessio Brancolini, and Vesa Julin, *Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality*, to appear, Annals of Probability, 2016.
- [Bob97] S. G. Bobkov, *An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in gauss space*, Ann. Probab. **25** (1997), no. 1, 206–214.
- [Bor85] Christer Borell, *Geometric bounds on the Ornstein-Uhlenbeck velocity process*, Z. Wahrsch. Verw. Gebiete **70** (1985), no. 1, 1–13. MR 795785 (87k:60103)
- [Bor03] ———, *The Ehrhard inequality*, C. R. Math. Acad. Sci. Paris **337** (2003), no. 10, 663–666. MR 2030108 (2004k:60102)
- [BS01] A. Burchard and M. Schmuckenschlger, *Comparison theorems for exit times*, Geometric & Functional Analysis GAFA **11** (2001), no. 4, 651–692 (English).
- [Cha17] Jui-En Chang, *1-dimensional solutions of the λ -self shrinkers*, Geometriae Dedicata (2017), 1–16.
- [CIMW13] Tobias Holck Colding, Tom Ilmanen, William P. Minicozzi, II, and Brian White, *The round sphere minimizes entropy among closed self-shrinkers*, J. Differential Geom. **95** (2013), no. 1, 53–69. MR 3128979
- [CM12] Tobias H. Colding and William P. Minicozzi, II, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) **175** (2012), no. 2, 755–833. MR 2993752
- [CR11] Amit Chakrabarti and Oded Regev, *An optimal lower bound on the communication complexity of gap Hamming distance*, Proc. 43rd Annual ACM Symposium on the Theory of Computing, 2011, pp. 51–60.
- [CW14] Qing-Ming Cheng and Guoxin Wei, *The gauss image of λ -hypersurfaces and a bernstein type problem*, Preprint, arXiv:1410.5302, 2014.
- [EH89] Klaus Ecker and Gerhard Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math. (2) **130** (1989), no. 3, 453–471. MR 1025164
- [Eld15] Ronen Eldan, *A two-sided estimate for the gaussian noise stability deficit*, Inventiones mathematicae **201** (2015), no. 2, 561–624 (English).
- [Gua] Qiang Guang, *Gap and rigidity theorems of λ -hypersurfaces*, Preprint, arXiv:1405.4871.
- [Hei15] Steven Heilman, *Low correlation noise stability of symmetric sets*, Preprint, arXiv:1511.00382, 2015.
- [Hui90] Gerhard Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), no. 1, 285–299. MR 1030675
- [Hui93] ———, *Local and global behaviour of hypersurfaces moving by mean curvature*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 175–191. MR 1216584
- [Led94] Michel Ledoux, *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space*, Bull. Sci. Math. **118** (1994), no. 6, 485–510. MR 1309086 (96c:49061)
- [Led96] ———, *Isoperimetry and Gaussian analysis*, Lectures on probability theory and statistics (Saint-Flour, 1994), Lecture Notes in Math., vol. 1648, Springer, Berlin, 1996, pp. 165–294. MR 1600888 (99h:60002)
- [Lee03] John M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, New York, 2003. MR 1930091 (2003k:58001)

- [Man17] Domenico Angelo La Manna, *Local minimality of the ball for the Gaussian perimeter*, Preprint,, 2017.
- [MN15a] Elchanan Mossel and Joe Neeman, *Robust optimality of Gaussian noise stability*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 2, 433–482. MR 3317748
- [MN⁺15b] Elchanan Mossel, Joe Neeman, et al., *Robust dimension free isoperimetry in gaussian space*, The Annals of Probability **43** (2015), no. 3, 971–991.
- [MR15] Matthew McGonagle and John Ross, *The hyperplane is the only stable, smooth solution to the isoperimetric problem in Gaussian space*, Geom. Dedicata **178** (2015), 277–296. MR 3397495
- [O'D12] Ryan O'Donnell, *Open problems in analysis of boolean functions*, Preprint, arXiv:1204.6447v1, 2012.
- [SC74] V. N. Sudakov and B. S. Cirel'son, *Extremal properties of half-spaces for spherically invariant measures*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **41** (1974), 14–24, 165, Problems in the theory of probability distributions, II. MR 0365680 (51 #1932)
- [Wan11] Lu Wang, *A Bernstein type theorem for self-similar shrinkers*, Geom. Dedicata **151** (2011), 297–303. MR 2780753
- [Zhu16] Jonathan J. Zhu, *On the entropy of closed hypersurfaces and singular self-shrinkers*, Preprint, arXiv:1607.07760, 2016.

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