Please provide complete and well-written solutions to the following exercises.

No due date, but the quiz in Week 8 in the discussion section (on October 18) will be based upon this homework.

Q8: Quiz 8 Problems

Exercise 1. State whether or not the statement is True or False. Justify your answer. Let $a < b$.

(1) If $f, g\colon [a, b] \to \mathbf{R}$ are continuous, then
$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$
(2) If $f, g\colon [a, b] \to \mathbf{R}$ are continuous, then
$$\int_a^b (f(x)g(x))dx = \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right).$$
(3) If $f, g\colon [a, b] \to \mathbf{R}$ are continuous, and if $f(x) \geq g(x)$ for all $x \in [a, b]$, then
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$
(4) If $f\colon (a, b) \to \mathbf{R}$ is continuous, then $\int_a^b f(x)dx$ exists.

Exercise 2. Suppose we are manufacturing n shoes. The average cost $AC(n)$ of producing n shoes is the total cost of producing all n shoes, divided by n. Assume that
$$AC(n) = \frac{100}{n} + 50 + 5n^2.$$

- What is the total cost $C(n)$ of producing n shoes?
- What is the fixed cost? (The fixed cost is the constant term in $C(n)$, i.e. the part of $C(n)$ that does not depend on n.)
- What is the marginal cost?

Exercise 3. Suppose $R(t)$ is a (continuous) stream of income at any time $t \geq 0$ going into a bank account. If the bank account earns interest rate r, then after T years, the account has
$$\int_0^T R(t)e^{rt}dt$$
dollars in it.

- Suppose $r = .03$ and $R(t) = 5000$. How much money is in the account after $T = 10$ years?

- Suppose $r = .03$ and R is defined by
$$R(t) = \begin{cases} 4000 & , 0 \le t \le 3 \\ 6000 & , 3 < t \le 6 \\ 6000e^{.03(t-6)} & , 6 < t \le 10. \end{cases}$$
How much money is in the account after $T = 10$ years?

The first case corresponds to putting money in the account at a roughly constant rate, whereas the second case corresponds to increasing your deposits into the account over time.

In each case, what is the present value of the money that is in the account ten years from now?

Exercise 4. A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate $f = f(t) \ge 0$, where t is the time measured in months since its last overhaul. Since a fixed cost $A > 0$ is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.

(a) Explain why $\int_0^t f(s)ds$ represents the loss in value of the machine over the period of time t since the last overhaul.
(b) Let $C = C(t)$ be given by
$$C(t) = \frac{1}{t}\left[A + \int_0^t f(s)ds\right].$$
What does C represent and why would the company want to minimize C?
(c) Assume that $\lim_{s\to\infty} f(s) = \infty$. Show that C has a minimum value at one of the numbers $t = T$ where $C(T) = f(T)$.

Exercise 5. A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate $f = f(t)$ and will accumulate maintenance costs at the rate $g = g(t)$, where t is the time measure in months. The company wants to determine the optimal time to replace the system.

(a) Let
$$C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]ds.$$
Show that the critical points of C occur at the numbers t where $C(t) = f(t) + g(t)$.
(b) Suppose
$$f(t) = \begin{cases} \frac{V}{15} - \frac{V}{450}t & , \text{if } 0 < t \le 30 \\ 0 & , \text{if } t > 30 \end{cases},$$
and suppose $g(t) = \frac{Vt^2}{12900}$ for $t > 0$. Determine the length of time T for the total depreciation $D(t) = \int_0^t f(s)ds$ to equal the initial value V.
(c) Determine the absolute minimum of C on $(0, T]$.
(d) Sketch the graphs of C and $f + g$ in the same coordinate system, and verify the result of part (a) in this case.

Exercise 6. Using integration by substitution, compute the following integrals

- $\int te^{t^2} dt$.
- $\int_0^1 x^3(x^4+1)^6 dx$.

Exercise 7. Using integration by parts, compute the following integrals

- $\int xe^x dx$.
- $\int_2^4 x(\ln x)^2 dx$.