Please provide complete and well-written solutions to the following exercises.

Due November 15, in the discussion section.

Homework 6

Exercise 1. Using any method you want to use, minimize

$$x_2 - x_1$$

subject to the constraints

$$x_1 \geq 0,\, x_2 \geq 0,\, x_1 + x_2 \leq 2,\, x_2 \leq 1.$$

(Note that only drawing a picture is insufficient justification.)

Exercise 2. Show that the dual problem of the dual problem is the primal problem. (Hint: first write the constraint $A^T y \leq c$ as $A^T y + z = c$, $z \geq 0$.)

Exercise 3. Show that the intersection of two closed sets is a closed set.

Exercise 4. Let $b \in \mathbf{R}^n$. Define $f\colon \mathbf{R}^n \to \mathbf{R}$ by $f(x) := ||x - b||$. Show that f is continuous. (Hint: show that it suffices to consider the case $b = 0$. In the case $b = 0$, use a triangle inequality.)

Exercise 5 (Farkas' Lemma, Version 2)**.** Let A be an $m \times n$ real matrix and let $b \in \mathbf{R}^m$. Then there exists $x \in \mathbf{R}^n$, such that $Ax \leq b$ if and only if: for all $y \in \mathbf{R}^m$ with $y \geq 0$ and $y^T A = 0$, we have $y^T b \geq 0$.

(Hint: $Ax \leq b$ has a solution if and only if $Ax^+ - Ax^- + z = b$, where $x^+, x^-, z \geq 0$. Apply Farkas' Lemma, Version 1, to the equality.)

Exercise 6. Give an example of a linear program where both the primal and dual problems are infeasible.

Exercise 7. Consider the linear program

$$\text{maximize} \quad y_1 \quad \text{subject to the constraints}$$

$$y_1 \geq 0,\, y_2 \geq 0,\, y_1 + y_2 \leq 1.$$

Draw the feasible region, and draw the point where the maximum occurs.

Find the dual problem, draw the feasible region for the dual problem, and draw the point where the minimum occurs.

Exercise 8. Using the Strong Duality for Linear Programming, prove von Neumann's Minimax Theorem from Game Theory:

Let m, n be positive integers. Let A be an $m \times n$ real matrix. Define

$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ \forall\, 1 \leq i \leq m\}.$$

Then

$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T A y = \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T A y.$$

(Hint: consider the linear program of maximizing t subject to the constraints: $t - \sum_{j=1}^{n} a_{ij} y_j \leq 0$ for all $1 \leq i \leq m$, $\sum_{j=1}^{n} y_j \leq 1$, $y \geq 0$.)

(Hint: First show that $\max_{x \in \Delta_m} x^T A y = \max_{i=1,\ldots,m} (Ay)_i = \max_{i=1,\ldots,m} \sum_{j=1}^{n} a_{ij} y_j$. So, the linear program mentioned above should compute the term on the right side of the equality.)