Math 167, Winter 2016, UCLA — Instructor: Steven Heilman

Name: ______________ UCLA ID: ______________ Date: ______________

Signature: ______________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	15	
2	15	
3	10	
4	15	
5	15	
Total:	70	

Reference sheet

Below are some definitions that may be relevant.

$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^{m} x_i = 1,\, x_i \geq 0,\ \forall\, 1 \leq i \leq m\}.$$

Let m, n be positive integers. Suppose we have a two-player general sum game with $m \times n$ payoff matrices. Let A be the payoff matrix for player I and let B be the payoff matrix for player II. A pair of vectors $(\widetilde{x}, \widetilde{y})$ with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$ is a **Nash equilibrium** if

$$\widetilde{x}^T A \widetilde{y} \geq x A \widetilde{y}, \quad \forall\, x \in \Delta_m,$$

$$\widetilde{x}^T B \widetilde{y} \geq \widetilde{x} B y, \quad \forall\, y \in \Delta_n.$$

A joint distribution of strategies is an $m \times n$ matrix $z = (z_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ such that $z_{ij} \geq 0$ for all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, and such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} = 1.$$

We say z is a **correlated equilibrium** if

$$\sum_{j=1}^{n} z_{ij} a_{ij} \geq \sum_{j=1}^{n} z_{ij} a_{kj}, \qquad \forall\, i \in \{1, \ldots, m\}, \forall\, k \in \{1, \ldots, m\}.$$

$$\sum_{i=1}^{m} z_{ij} b_{ij} \geq \sum_{i=1}^{m} z_{ij} b_{ik}, \qquad \forall\, j \in \{1, \ldots, n\}, \forall\, k \in \{1, \ldots, n\}.$$

1. (15 points) Recall the prisoner's dilemma, which has the following payoffs.

Prisoner I	Prisoner II: silent	Prisoner II: confess
silent	$(-1, -1)$	$(-10, 0)$
confess	$(0, -10)$	$(-8, -8)$

Find all Nash equilibria for this game.

2. (15 points) Find the value of the two-person zero-sum game described by the payoff matrix

$$\begin{pmatrix} 3 & 1 & 2 & 0 \\ 4 & 0 & 5 & 3 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$

3. (10 points) Let $f\colon [0,1] \to [0,1]$ be a continuous function. Prove that there exists $x \in [0,1]$ such that $f(x) = x$. (That it, prove the Brouwer Fixed Point Theorem in dimension 1. That is, you cannot just say this follows from the Brouwer Fixed Point Theorem, you have to prove it.) (Hint: you can freely use the Intermediate Value Theorem.)

4. (a) (5 points) Give an example of a closed and convex subset K of Euclidean space, and give an example of a continuous function $f\colon K \to K$ such that f has no fixed point. (That is, for every $x \in K$, $f(x) \neq x$.)

   (b) (5 points) Give an example of a convex and bounded subset K of Euclidean space, and give an example of a continuous function $f\colon K \to K$ such that f has no fixed point.

   (c) (5 points) Give an example of a function $f\colon [0,1] \to [0,1]$ such that f has no fixed point.

5. (15 points) Prove that any Nash equilibrium is a Correlated Equilibrium. (That is, if m, n are positive integers, and if $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$, then $\widetilde{x}\widetilde{y}^T$ is a correlated equilibrium.) (Here we regard $\widetilde{x}$ and $\widetilde{y}$ as column vectors.)

(Scratch paper)