Math 167, Winter 2016, UCLA | Instructor: Steven Heilman

Name: ____________________ UCLA ID: _______________ Date: ____________

Signature: ____________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Final Exam

This exam contains 16 pages (including this cover page) and 11 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 180 minutes to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	15	
2	15	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	15	
11	10	
Total:	125	

Reference sheet

Below are some definitions that may be relevant.

$$\Delta_m := \{x = (x_1, \ldots, x_m) \in \mathbf{R}^m \colon \sum_{i=1}^{m} x_i = 1,\, x_i \geq 0,\ \forall\, 1 \leq i \leq m\}.$$

Let m, n be positive integers. Suppose we have a two-player general sum game with $m \times n$ payoff matrices. Let A be the payoff matrix for player I and let B be the payoff matrix for player II. A pair of vectors $(\widetilde{x}, \widetilde{y})$ with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$ is a **Nash equilibrium** if

$$\widetilde{x}^T A \widetilde{y} \geq x A \widetilde{y}, \quad \forall\, x \in \Delta_m,$$
$$\widetilde{x}^T B \widetilde{y} \geq \widetilde{x} B y, \quad \forall\, y \in \Delta_n.$$

A joint distribution of strategies is an $m \times n$ matrix $z = (z_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ such that $z_{ij} \geq 0$ for all $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, and such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} = 1.$$

We say z is a **correlated equilibrium** if

$$\sum_{j=1}^{n} z_{ij} a_{ij} \geq \sum_{j=1}^{n} z_{ij} a_{kj}, \qquad \forall\, i \in \{1, \ldots, m\}, \forall\, k \in \{1, \ldots, m\}.$$

$$\sum_{i=1}^{m} z_{ij} b_{ij} \geq \sum_{i=1}^{m} z_{ij} b_{ik}, \qquad \forall\, j \in \{1, \ldots, n\}, \forall\, k \in \{1, \ldots, n\}.$$

Suppose we have a two-player symmetric game (so that the payoff matrix for player I is A, the payoff matrix for player II is B, and with $A = B^T$). Assume that A, B are $n \times n$ matrices. A mixed strategy $x \in \Delta_n$ is said to be an **evolutionarily stable strategy** if, for any pure strategy w, we have

$$w^T A x \leq x^T A x,$$
$$\text{If } w^T A x = x^T A x, \text{ then } w^T A w < x^T A w.$$

Suppose we have a game with n players together with a characteristic function $v \colon 2^{\{1, \ldots, n\}} \to \mathbf{R}$. For each $i \in \{1, \ldots, n\}$, we define the **Shapley value** $\phi_i(v) \in \mathbf{R}$ to be any set of real numbers satisfying the following four axioms:

(i) (Symmetry) If for some $i, j \in \{1, \ldots, n\}$ we have $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq \{1, \ldots, n\}$ with $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$.

(ii) (No power/ no value) If for some $i \in \{1, \ldots, n\}$ we have $v(S \cup \{i\}) = v(S)$ for all $S \subseteq \{1, \ldots, n\}$, then $\phi_i(v) = 0$.

(iii) (Additivity) If u is any other characteristic function, then $\phi_i(v + u) = \phi_i(v) + \phi_i(u)$, for all $i \in \{1, \ldots, n\}$.

(iv) (Efficiency) $\sum_{i=1}^{n} \phi_i(v) = v(\{1, \ldots, n\})$.

We define a **symmetric auction**. A single object is for sale at an auction. The seller is willing to sell the object at any nonnegative price. There are n buyers, which we identify with the set $\{1, 2, \ldots, n\}$. All buyers have some set of **private values** in $[0, 1]$. We denote the private value of buyer $i \in \{1, \ldots, n\}$ by V_i, so that V_i is a random variable that takes values in $[0, 1]$. We assume that all of the random variables $V_1, \ldots, V_n$ are independent. We also assume that $V_1, \ldots, V_n$ are identically distributed, with a continuous density function. That is, there exists some continuous function $f\colon [0, 1] \to [0, \infty)$ with $\int_0^1 f(x)dx = 1$ such that: for each $i \in \{1, \ldots, n\}$, for each $t \in [0, 1]$, the probability that $V_i \leq t$ is equal to $\int_0^t f(x)dx$. We define the **expected value** of V_1 to be $\int_0^1 xf(x)dx$. Finally, we assume that all buyers are **risk-neutral**, so that each buyer seeks to maximize her expected profits.

Finally, we assume that all of the above assumptions are **common knowledge**. That is, every player knows the above assumptions; every player knows that every player knows the above assumptions; every player knows that every player knows that every player knows the above assumptions; etc.

Under the above assumptions, a **pure strategy** for Player $i \in \{1, \ldots, n\}$ is a function $\beta_i\colon [0, 1] \to [0, \infty)$. So, if Player i has a private value of V_i, he will make a bid of $\beta_i(V_i)$ in the auction. (We will not discuss mixed strategies in auctions.)

Given the strategies $\beta = (\beta_1, \ldots, \beta_n)$, and given any $v \in [0, 1]$, Player i has expected profit $P_i(\beta, v)$, if her private value is v. (If buyer i wins the auction, and if buyer i has private value v and bid b, then the profit of buyer i is $v - b$.) We say that a strategy β is an **equilibrium** if, given any $v \in [0, 1]$, any $b \geq 0$, and any $i \in \{1, \ldots, n\}$,

$$P_i(\beta, v) \geq P_i((\beta_1, \ldots, \beta_{i-1}, b, \beta_{i+1}, \ldots, \beta_n), v).$$

Let n be a positive integer. Let $x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$. Let $f, g\colon \{-1, 1\}^n \to \mathbf{R}$. For any subset $S \subseteq \{1, \ldots, n\}$, define a function $W_S\colon \{-1, 1\}^n \to \mathbf{R}$ by $W_S(x) := \prod_{i \in S} x_i$. Define also the inner product $\langle f, g\rangle := 2^{-n} \sum_{x \in \{-1,1\}^n} f(x)g(x)$. Any $f\colon \{-1, 1\}^n \to \mathbf{R}$ can be expressed as $f(x) = \sum_{S \subseteq \{1,\ldots,n\}} \langle f, W_S\rangle W_S(x)$. For any $S \subseteq \{1, \ldots, n\}$, if we denote $\widehat{f}(S) := \langle f, W_S\rangle = 2^{-n} \sum_{y \in \{-1,1\}^n} f(y)W_S(y)$, then we have $f(x) = \sum_{S \subseteq \{1,\ldots,n\}} \widehat{f}(S)W_S(x)$.

The **noise stability** of f with parameter $\rho \in (-1, 1)$ is defined to be $\sum_{S \subseteq \{1,\ldots,n\}} \rho^{|S|} |\widehat{f}(S)|^2$.

1. Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning. (In this question, you can freely cite results from the homeworks.)

   (a) (3 points) Every two-player general sum game has a Nash equilibrium such that this Nash equilibrium is evolutionarily stable.

   TRUE FALSE (circle one)

   (b) (3 points) Every two-player general sum game has at least two correlated equilibria.

   TRUE FALSE (circle one)

   (c) (3 points) There exists a symmetric two-person general-sum game such that all of its Nash equilibria are not symmetric.

   TRUE FALSE (circle one)

   (d) (3 points) Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a function. Suppose we do a Condorcet election with f (so that if we just look at the votes between any pair of two candidates, the aggregate preference is decided using the function f). Suppose a Condorcet winner always exists. Then there exists $i \in \{1,\ldots,n\}$ such that $f(x_1,\ldots,x_n) = x_i$ for all $(x_1,\ldots,x_n) \in \{-1,1\}^n$.

   TRUE FALSE (circle one)

   (e) (3 points) Let $f, g\colon \{-1,1\}^n \to \{-1,1\}$ be functions. Assume that n is odd and f is the majority function ($f(x_1,\ldots,x_n) = \mathrm{sign}(x_1+\cdots+x_n)$). Assume $f \neq g$. Then for any $\rho \in (0,1)$, the noise stability of f exceeds the noise stability of g.

   TRUE FALSE (circle one)

2. (15 points) Consider the two-person zero-sum game defined by the following payoff matrix

		Player II								
		1	2	3	4	5	6	$\cdots$		n
Player I	1	0	-1	2	2	2	2	$\cdots$		2
	2	1	0	-1	2	2	2	$\cdots$		2
	3	-2	1	0	-1	2	2	$\cdots$		2
	4	-2	-2	1	0	-1	2	$\cdots$		2
	5	-2	-2	-2	1	0	-1	$\cdots$		2
	$\vdots$	$\vdots$	$\vdots$					$\ddots$		$\vdots$
	$n-1$	-2	-2	$\cdots$				1	0	-1
	n	-2	-2	$\cdots$					1	0

Compute the value of the game. Also, find at least one pair of optimal strategies for both players.

3. (10 points) Recall the Game of Chicken has the following payoff matrix

Player I \ Player II	C	D
C	$(6, 6)$	$(2, 7)$
D	$(7, 2)$	$(0, 0)$

Find all Nash equilibria for the Game of Chicken. Prove that these are the only Nash equilibria.

4. (10 points) Prove the following generalization of Brouwer's fixed point theorem:

   Let K be a convex, closed, bounded subset of Euclidean space $\mathbf{R}^n$. Let L be any subset of Euclidean space $\mathbf{R}^n$. Suppose there exist continuous functions $S\colon K \to L$ and $T\colon L \to K$ such that $(ST)(x) = x$ for all $x \in L$ and $(TS)(x) = x$ for all $x \in K$. Let $f\colon L \to L$ be continuous. Show that f has at least one fixed point. That is, there exists some $x \in L$ with $f(x) = x$. (Hint: apply Brouwer's fixed point theorem to K.)

5. (10 points) Prove that any Nash equilibrium is a Correlated Equilibrium. (That is, if m, n are positive integers, and if $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$, then $\widetilde{x}\widetilde{y}^T$ is a correlated equilibrium.) (Here we regard $\widetilde{x}$ and $\widetilde{y}$ as column vectors.)

6. (10 points) Suppose we have a two person general sum game. Let K be the set of all Correlated equilibria for the game. Prove that K is a convex set.

7. (10 points) Define $v\colon 2^{\{1,2,3\}} \to \mathbf{R}$ so that $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$, whereas $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\emptyset) = 0$.

   Arguing directly using **the axioms for the Shapley value**, compute all of the Shapley values of v.

8. (10 points) Ten people are standing together in a room. They are presented with the following problem: each person chooses a real number between (and including) 0 and 100. (So, someone could guess: 20, 51.5, π, $\sqrt{2}$, 99.999, etc.) The person who chooses the number closest to two-thirds of the average of all of the numbers wins. That is, if the numbers are $0 \leq a_1, \ldots, a_{10} \leq 100$, each person wants to choose a number closest to $\frac{2}{3} \cdot \frac{1}{10} \sum_{i=1}^{10} a_i$. The people do not communicate with each other in any way. The rules of the game are common knowledge. It is common knowledge that every person wants to win the game, and every person is rational. Explain what number each person will choose.

9. (10 points) Suppose we have two buyers, and $f(v) = 1$ for any $v \in [0, 1]$ in a sealed-bid second price auction. That is, the private values V_1 and V_2 are uniformly distributed in the interval $[0, 1]$. Show that an equilibrium strategy is $\beta_1(v) = v$, $\beta_2(v) = v$, $\forall\, v \in [0, 1]$. That is, each player will bid exactly her private value.

   (In a sealed-bid second price auction, every buyer submits a sealed envelope with her desired bid for the item. The buyer who has submitted the highest bid receives the item for the *second-highest bid*.)

10. (15 points) Let $f\colon \{-1,1\}^n \to \{-1,1\}$. Assume that $\widehat{f}(S) = 0$ whenever $S \subseteq \{1,\ldots,n\}$ and $|S| \neq 1$. Show that there exists $i \in \{1,\ldots,n\}$ such that $f(x) = f(x_1,\ldots,x_n) = x_i$ for all $x \in \{-1,1\}^n$, or $f(x) = -x_i$ for all $x \in \{-1,1\}^n$. (Recall that $|S|$ denotes the number of elements of S.)

11. (10 points) Explain in detail the Condorcet voting paradox. You should probably use an example of three voters ranking three candidates in order to explain the paradox.

(Scratch paper)

(More scratch paper)