167 Final Solutions[1]

1. QUESTION 1

(a) Every two-player general sum game has a Nash equilibrium such that this Nash equilibrium is evolutionarily stable.

False. Rock paper scissors has only one nash equilibrium, and it is not evolutionarily stable. (We showed this on the homework.)

(b) Every two-player general sum game has at least two correlated equilibria.

False. The Prisoner's Dilemma has only one correlated equilibrium. (We showed this on the homework.)

(c) There exists a symmetric two-person general-sum game such that all of its Nash equilibria are not symmetric.

False. Every symmetric game has at least one symmetric Nash equilibrium. (This was a Corollary of Nash's Theorem, or more precisely, it follows by repeating the proof of Nash's theorem.)

(d) Let $f\colon \{-1,1\}^n \to \{-1,1\}$ be a function. Suppose we do a Condorcet election with f (so that if we just look at the votes between any pair of two candidates, the aggregate preference is decided using the function f). Suppose a Condorcet winner always exists. Then there exists $i \in \{1,\ldots,n\}$ such that $f(x_1,\ldots,x_n) = x_i$ for all $(x_1,\ldots,x_n) \in \{-1,1\}^n$.

False. it could occur that $f(x_1,\ldots,x_n) = -x_i$ for all $(x_1,\ldots,x_n) \in \{-1,1\}^n$.

(e) Let $f,g\colon \{-1,1\}^n \to \{-1,1\}$ be functions. Assume that n is odd and f is the majority function $(f(x_1,\ldots,x_n) = \mathrm{sign}(x_1 + \cdots + x_n))$. Assume $f \neq g$. Then, for any $\rho \in (0,1)$, the noise stability of f exceeds the noise stability of g.

False. The constant function $f = 1$ has the largest noise stability among all such functions (which is 1).

2. QUESTION 2

Consider the two-person zero-sum game defined by the following payoff matrix

Player II

		1	2	3	4	5	6	$\cdots$	n
Player I	1	0	-1	2	2	2	2	$\cdots$	2
	2	1	0	-1	2	2	2	$\cdots$	2
	3	-2	1	0	-1	2	2	$\cdots$	2
	4	-2	-2	1	0	-1	2	$\cdots$	2
	5	-2	-2	-2	1	0	-1	$\cdots$	2
	$\vdots$	$\vdots$	$\vdots$					$\ddots$	$\vdots$
	$n-1$	-2	-2	$\cdots$				1	0	-1
	n	-2	-2	$\cdots$				1	0

Compute the value of the game. Also, find at least one pair of optimal strategies for both players.

Solution. The value is zero, and the optimal strategy is $(1/4, 1/2, 1/4, 0, \ldots, 0)$ for both players. each row with index at least 4 is dominated by the first row. That is, $(Ay)_1 \geq (Ay)_i$ for any $4 \leq i \leq n$. From an Exercise from class, the value of the game is equal to

$$\min_{y \in \Delta_n} \max_{x \in \Delta_n} x^T A y = \min_{y \in \Delta_n} \max_{i=1,\ldots,n} (Ay)_i = \min_{y \in \Delta_n} \max_{i=1,2,3} (Ay)_i.$$

That is, for the purpose of computing the value of the game, we can ignore all rows except for the first three. Similarly, each column with index at least 4 dominates the first column. That is, $(x^T A)_1 \leq (x^T A)_j$ for any $4 \leq j \leq n$. From the Exercise mentioned before, the value of the game is equal to

$$\max_{x \in \Delta_n} \min_{y \in \Delta_n} x^T A y = \max_{x \in \Delta_n} \min_{j=1,\ldots,n} (x^T A)_j = \max_{x \in \Delta_m} \min_{j=1,2,3} (x^T A)_j.$$

That is, for the purpose of computing the value of the game, we can ignore all columns except for the first three.

In summary, the value of the original game has the same value as the following game

Player II

	1	2	3
1	0	-1	2
2	1	0	-1
3	-2	1	0

(Player I labels the rows)

Let A denote this 3×3 payoff matrix. Note that $A = -A^T$. That is, A is antisymmetric. We first claim that the value of the game corresponding to A is zero. To see this, note that the value is

$$\min_{y \in \Delta_3} \max_{x \in \Delta_3} x^T A y = \min_{y \in \Delta_3} \max_{x \in \Delta_3} x^T (-A^T) y = \min_{y \in \Delta_3} \max_{x \in \Delta_3} -(y^T A x)^T = \min_{y \in \Delta_3} \left[-\min_{x \in \Delta_3} y^T A x \right]$$

$$= -\max_{y \in \Delta_3} \min_{x \in \Delta_3} y^T A x = -\min_{x \in \Delta_3} \max_{y \in \Delta_3} y^T A x = -\min_{y \in \Delta_3} \max_{x \in \Delta_3} x^T A y.$$

In the penultimate equality, we used von Neumann's Minimax Theorem. So, the value is equal to the negative of itself, so it must be zero. Now, using an exercise from the notes, the value is

$$0 = \max_{x \in \Delta_3} \min_{j=1,2,3} (x^T A)_j = \max_{x \in \Delta_3} \min(x_2 - 2x_3, -x_1 + x_3, 2x_1 - x_2).$$

So, if $x \in \Delta_3$ is the optimal strategy for Player I, we have $\min(x_2 - 2x_3, -x_1 + x_3, 2x_1 - x_2) = 0$, so that $(x_2 - 2x_3, -x_1 + x_3, 2x_1 - x_2) \geq (0, 0, 0)$, i.e. $x_2 \geq 2x_3$, $2x_3 \geq 2x_1$ and $2x_1 \geq x_2$. If any of these inequalities were strict, their combination would say $x_2 > x_2$, which is false. So, each of these inequalities must be an equality. That is, $x_2 = 2x_3 = 2x_1 = x_2$. Since $x_1 + x_2 + x_3 = 1$, we conclude that $x_3 = x_1 = 1/4$ and $x_2 = 1/2$. So, the optimal strategy for player I is $(1/4, 1/2, 1/4)$.

Using similar reasoning for y, the optimal strategy for player II is also $(1/4, 1/2, 1/4)$.

3. Question 3

Recall the Game of Chicken has the following payoff matrix
Find all Nash equilibria for the Game of Chicken. Prove that these are the only Nash equilibria.

Player II

	C	D
C	$(6,6)$	$(2,7)$
D	$(7,2)$	$(0,0)$

(Player I labels the rows.)

Solution. Let (x, y) be a Nash equilibrium. Write $x = (x_1, x_2) = (s, 1 - s)$, $y = (y_1, y_2) = (t, 1 - t)$ where $s, t \in [0, 1]$. Then

$$x^T A y = 6x_1 y_1 + 2x_1 y_2 + 7x_2 y_1 = 6st + 2s(1 - t) + 7(1 - s)t = -3st + 2s + 7t.$$

Let $f(s, t) = -3st + 2s + 7t$. Then $\partial f / \partial s = -3t + 2$. So, if $t = 2/3$, f is maximized for any s. If $t > 2/3$, $f(s, t)$ is maximized over s by $s = 0$. If $t < 2/3$, $f(s, t)$ is maximized over s by $s = 1$.

We split into three cases: either $t = 2/3$, $t > 2/3$ (so that $s = 0$, by definition of Nash equilibrium), or $t < 2/3$ (so that $s = 1$, by definition of Nash equilibrium).

$$x^T B y = 6x_1 y_1 + 7x_1 y_2 + 2x_2 y_1 = 6st + 7s(1 - t) + 2(1 - s)t = -3st + 7s + 2t.$$

Let $g(s, t) = -3st + 7s + 2t$. If $t > 2/3$, then $s = 0$, so $g(s, t) = 2t$ which is maximized over t only when $t = 1$. So, $(0, 1)$, $(1, 0)$ is a Nash equilibrium. If $t < 2/3$, then $s = 1$, so $g(s, t) = -3t + 7$ which is maximized over t only when $t = 0$. So, $(1, 0)$, $(0, 1)$ is a Nash equilibrium. In the case $t = 2/3$, note that $\partial g / \partial t = -3s + 2$, which as above, splits into three cases. Two of the cases have already been discussed, and the remaining case occurs when $s = 2/3$. And we can verify that $(2/3, 1/3)$, $(2/3, 1/3)$ is the only remaining Nash equilibrium.

4. QUESTION 4

Prove the following generalization of Brouwer's fixed point theorem:

Let K be a convex, closed, bounded subset of Euclidean space $\mathbb{R}^n$. Let L be any subset of Euclidean space $\mathbb{R}^n$. Suppose there exist continuous functions $S \colon K \to L$ and $T \colon L \to K$ such that $S(T(x)) = x$ for all $x \in L$ and $T(S(x)) = x$ for all $x \in K$. Let $f \colon L \to L$ be continuous. Show that f has at least one fixed point. That is, there exists some $x \in L$ with $f(x) = x$. (Hint: apply Brouwer's fixed point theorem to K.)

Solution. Consider the function $TfS \colon K \to K$ which is continuous, since it is a composition of continuous functions. By Brouwer's Theorem, TfS has a fixed point. That is, there exists $y \in K$ such that $(TfS)(y) = y$. Applying S to both sides, we have $(fS)(y) = S(y)$, that is, $f(S(y)) = S(y)$. That is, $S(y)$ is a fixed point for f. And since $y \in K$ and $S \colon K \to L$, we have $S(y) \in L$, as desired.

5. QUESTION 5

Prove that any Nash equilibrium is a Correlated Equilibrium. (That is, if m, n are positive integers, and if $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium with $\widetilde{x} \in \Delta_m$ and $\widetilde{y} \in \Delta_n$, then $\widetilde{x}\widetilde{y}^T$ is a correlated equilibrium.) (Here we regard $\widetilde{x}$ and $\widetilde{y}$ as column vectors.)

Solution. We argue by contradiction. Suppose $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium. Let $z = \widetilde{x}\widetilde{y}^T$. Suppose for the sake of contradiction that z is not a correlated equilibrium. Then the

3

negation of the definition of correlated equilibrium holds. Without loss of generality, the negated condition applies to player I. That is, there exists $i, k \in \{1, \ldots, m\}$ such that

$$\sum_{j=1}^{n} z_{ij} a_{ij} < \sum_{j=1}^{n} z_{ij} a_{kj}.$$

That is,

$$\widetilde{x}_i \sum_{j=1}^{n} \widetilde{y}_j a_{ij} < \widetilde{x}_i \sum_{j=1}^{n} \widetilde{y}_j a_{kj}. \qquad (*)$$

This inequality suggests that Player I can benefit by switching from strategy i to strategy k in the mixed strategy $\widetilde{x}$. Let $e_i \in \Delta_m$ denote the vector with a 1 in the ith entry and zeros in all other entries. Define $x \in \Delta_m$ so that $x = \widetilde{x} - \widetilde{x}_i e_i + \widetilde{x}_i e_k$. Observe that

$$x^T A \widetilde{y} - \widetilde{x}^T A \widetilde{y} = (-\widetilde{x}_i e_i + \widetilde{x}_i e_k)^T A \widetilde{y} = -\widetilde{x}_i \sum_{j=1}^{n} a_{ij} \widetilde{y}_j + \widetilde{x}_i \sum_{j=1}^{n} a_{kj} \widetilde{y}_j \overset{(*)}{>} 0.$$

But this inequality contradicts that $(\widetilde{x}, \widetilde{y})$ is a Nash equilibrium.

6. Question 6

Suppose we have a two person general sum game. Let K be the set of all Correlated equilibria for the game. Prove that K is a convex set.

Solution. Let w, z be Correlated equilibria. Let $t \in [0, 1]$. We are required to show that $tz + (1-t)w$ is a Correlated equilbrium. Fix $i, k \in \{1, \ldots, m\}$. It is given than

$$\sum_{j=1}^{n} z_{ij} a_{ij} \geq \sum_{j=1}^{n} z_{ij} a_{kj}, \qquad \sum_{j=1}^{n} w_{ij} a_{ij} \geq \sum_{j=1}^{n} w_{ij} a_{kj}. \qquad (*)$$

So, using $t, (1-t) \geq 0$,

$$\sum_{j=1}^{n} (tz_{ij} + (1-t)w_{ij}) a_{ij} = t \sum_{j=1}^{n} z_{ij} a_{ij} + (1-t) \sum_{j=1}^{n} w_{ij} a_{ij}$$

$$\overset{(*)}{\geq} t \sum_{j=1}^{n} z_{ij} a_{kj} + (1-t) \sum_{j=1}^{n} w_{ij} a_{kj} = \sum_{j=1}^{n} (tz_{ij} + (1-t)w_{ij}) a_{kj}.$$

The analogous inequality holds for the matrix b_{ij} using the same argument. That is, $tz + (1-t)w$ is a Correlated equilibrium.

7. Question 7

Define $v \colon 2^{\{1,2,3\}} \to \mathbb{R}$ so that $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$, whereas $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\emptyset) = 0$.

Arguing directly using **the axioms for the Shapley value**, compute all of the Shapley values of v.

Solution. We first claim that $\phi_1(v) = \phi_2(v) = \phi_3(v)$. This will follow by an application of Axiom (i). We need to check the assumption of Axiom (i) holds for all eight subsets S of $\{1, 2, 3\}$. When $S = \emptyset$, it is given that $v(\{1\}) = v(\{2\}) = v(\{3\})$. When $S = \{1\}$, we know $v(\{1,2\}) = v(\{1,3\})$; similarly the assumption of Axiom (i) holds for $S = \{2\}$

and $S = \{3\}$. The remaining assumptions of Axiom (i) hold vacuously (in the case that $|S| \geq 2$). We conclude that Axiom (i) tells us $\phi_1(v) = \phi_2(v) = \phi_3(v)$. Now, from Axiom (iii), $\phi_1(v) + \phi_2(v) + \phi_3(v) = v(\{1, 2, 3\}) = 1$. In conclusion, $\phi_1(v) = \phi_2(v) = \phi_3(v) = 1/3$.

8. QUESTION 8

Ten people are standing together in a room. They are presented with the following problem: each person chooses a real number between (and including) 0 and 100. (So, someone could guess: 20, 51.5, π, $\sqrt{2}$, 99.999, etc.) The person who chooses the number closest to two-thirds of the average of all of the numbers wins. That is, if the numbers are $0 \leq a_1, \ldots, a_{10} \leq 100$, each person wants to choose a number closest to $(2/3)(1/10)\sum_{i=1}^{10} a_i$. The people do not communicate with each other in any way. It is common knowledge that every person wants to win the game, and every person is rational. Explain what number each person will choose.

Solution. Every person will choose 0.

To see why, consider the actions of the first person. Since every bid will be at most 100, we have

$$(2/3)(1/10)\sum_{i=1}^{10} a_i \leq (2/3)(1/10)\sum_{i=1}^{10} 100 = (2/3)100.$$

That is, the largest bid any person will rationally choose is $(2/3)100$. However, all players are aware of this fact, so all players know everyone will rationally bid at most $(2/3)100$. Consequently,

$$(2/3)(1/10)\sum_{i=1}^{10} a_i \leq (2/3)(1/10)\sum_{i=1}^{10}(2/3)(100) = (2/3)^2 100.$$

So, the largest bid any person will rationally choose is $(2/3)^2 100$. And all players are aware of this fact. And so on.

In general, if there is some upper bound T with $0 \leq T \leq 100$ on all guesses that all players will make, and if T is common knowledge, then the largest bid any rational person will consider making is

$$(2/3)(1/10)\sum_{i=1}^{10} a_i \leq (2/3)(1/10)\sum_{i=1}^{10} T = (2/3)T.$$

And this fact is also common knowledge. So, the largest bid any person will rationally make is $\lim_{N \to \infty}(2/3)^N T = 0$.

9. QUESTION 9

Suppose we have two buyers, and $f(v) = 1$ for any $v \in [0, 1]$ in a sealed-bid second price auction. That is, the private values V_1 and V_2 are uniformly distributed in the interval $[0, 1]$. Show that an equilibrium strategy is $\beta_1(v) = v$, $\beta_2(v) = v$, $\forall\, v \in [0, 1]$. That is, each player will bid exactly their private value.

Solution 1. Suppose buyer 2 uses this strategy, and suppose buyer 1 has private value $v \in [0, 1]$. Suppose buyer 1 submits the bid $b \in [0, 1]$. Since buyer 2 will bid V_2, buyer 1 wins the auction only when $b > V_2$. Since V_2 is uniformly distributed in $[0, 1]$, we have $b > V_2$ with probability $\int_0^b dx = b$.

The expected profit of buyer 1 is the expected value of $v - V_2$ multiplied by the function which is 1 when $b > V_2$ and 0 otherwise. That is, the expected profit of buyer 1 is

$$\int_0^b (v - x)dx = vb - b^2/2.$$

So, buyer 1 maximizes her profit by choosing $b = v$ (since the function $b \mapsto vb - b^2/2$ is maximized at $b = v$.). That is, we must have $\beta_1(v) = v$. Using a similar argument, if $\beta_1(v) = v$, then buyer 2 maximizes her profit by choosing $\beta_2(v) = v$. That is, the strategy $\beta_1(v) = v$, $\beta_2(v) = v$, $\forall\, v \in [0,1]$ is a symmetric equilibrium.

Solution 2. Suppose buyer 2 uses this strategy. Suppose buyer 1 has private value $v_1 \in [0,1]$. Suppose buyer 1 submits the bid $b \in [0,1]$, and suppose buyer 2 has private value $v_2 \in [0,1]$. Then buyer 2 bids v_2. We consider two different cases.

(Case 1) If $v_2 > v_1$, then buyer 1 can only win the auction when $b > v_2$, resulting in a profit of $v_1 - b \le v_1 - v_2 \le 0$. So, if $v_2 > v_1$, buyer 1 maximizes her profit by choosing b such that $b = v_1$ (achieving a profit of zero whether or not she wins the auction).

(Case 2) If $v_1 \ge v_2$, and if buyer 1 wins the auction (which occurs when $b > v_2$), then buyer 1 gets a positive profit of $v_1 - v_2$. So, if $v_1 \ge v_2$, buyer 1 maximizes her profit by choosing b such that $b = v_1$ (thereby winning the auction).

So, in any case, buyer 1 maximizes her profit by choosing $b = v_1$. That is, buyer 1 should choose $\beta_1(v) = v$ for all $v \in [0,1]$. A similar argument applies to buyer 2, so that $\beta_1(v) = v$, $\beta_2(v) = v$ is an equilibrium. (Note that this solution did not use any of the properties of the random variables V_1, V_2. That is, this strategy maximizes the profit for each player *before* taking any expectations, so this strategy also maximizes the profit for each player *after* taking expectations.)

10. Question 10

Let $f \colon \{-1,1\}^n \to \{-1,1\}$. Assume that $\widehat{f}(S) = 0$ whenever $S \subseteq \{1,\ldots,n\}$ and $|S| \ne 1$. Show that there exists $i \in \{1,\ldots,n\}$ such that $f(x) = f(x_1,\ldots,x_n) = x_i$ for all $x \in \{-1,1\}^n$, or $f(x) = -x_i$ for all $x \in \{-1,1\}^n$.

Solution. It is given that there exist $c_1,\ldots,c_n \in \mathbb{R}$ such that $f(x_1,\ldots,x_n) = \sum_{i=1}^n c_i x_i$ for all $(x_1,\ldots,x_n) \in \{-1,1\}^n$. We know that at least one of the numbers $c_1,\ldots,c_n$ is nonzero, since if all of them were zero, then $f(1,\ldots,1)$ would be zero as well, contradicting our assumption that $|f(1,\ldots,1)| = 1$.

For any $x \in \mathbb{R}$, define $\operatorname{sign}(x) = 1$ if $x > 0$, $\operatorname{sign}(x) = -1$ if $x \le 0$. Let $v = (\operatorname{sign}(c_1),\ldots,\operatorname{sign}(c_n))$. Then $f(v) = \sum_{i=1}^n c_i \operatorname{sign}(c_i) = \sum_{i=1}^n |c_i| > 0$ (since at least one of the numbers $c_1,\ldots,c_n$ is nonzero.) Also, $v \in \{-1,1\}^n$. Therefore, $f(v) = 1$. That is, $\sum_{i=1}^n |c_i| = 1$

Also, by Plancherel's Theorem (from Section 7.1 in the notes), $1 = \sum_{S \subseteq \{1,\ldots,n\}} |\widehat{f}(S)|^2 = \sum_{i=1}^n c_i^2$. In particular, $|c_i| \le 1$ for all $i \in \{1,\ldots,n\}$

In summary, $\sum_{i=1}^n |c_i| = 1$ and $\sum_{i=1}^n |c_i|^2 = 1$. But $|c_i| \le 1$ for all $i \in \{1,\ldots,n\}$, so $|c_i|^2 \le |c_i|$ for all $i \in \{1,\ldots,n\}$, with equality if and only if $|c_i| = 1$. So $\sum_{i=1}^n |c_i|^2 \le \sum_{i=1}^n |c_i|$, with equality only if $|c_j| = 1$ for some $j \in \{1,\ldots,n\}$. Since $\sum_{i=1}^n |c_i|^2 = \sum_{i=1}^n |c_i|$, we conclude that there must be some $j \in \{1,\ldots,n\}$ with $|c_j| = 1$, and the other c_i are zero, as desired.

11. Question 11

Explain in detail the Condorcet voting paradox. You should probably use an example of three voters ranking three candidates in order to explain the paradox.

Solution. Consider the following ranking of three candidates a, b, c between three voters $1, 2, 3$.

Voter	Rank 1	Rank 2	Rank 3
1	a	b	c
2	b	c	a
3	c	a	b

If we ignore candidate b, then voters 2 and 3 prefer c over a, while voter 1 prefers a over c. So, using a majority rule for these preferences, the voters prefer c over a. If we ignore candidate c, then voters 1 and 3 prefer a over b, while voter 2 prefers b over a. So, using a majority rule again, the voters prefer a over b. Finally, if we ignore candidate a, then voters 1 and 2 prefer b over c, while voter 3 prefers c over b. So, using a majority rule, the voters prefer b over c.

In conclusion, given the above ranking, if we use a majority rule for every comparison between two candidates, the voters prefer a over b, they prefer b over c, and they prefer c over a. So, no one has won the election, if we conduct a Condorcet election.