170A Practice Midterm 1 Solutions, Winter 2014[1]

1. Question 1

(a) If events A, B satisfy $\mathbf{P}(A) = .5$ and $\mathbf{P}(A \cap B) = .2$, find $\mathbf{P}(B|A)$.

Using the definition of conditional probability, we have $\mathbf{P}(B|A) = \mathbf{P}(B \cap A)/\mathbf{P}(A) = .2/.5 = 2/5$.

(b) What is $\mathbf{P}(\emptyset)$?

From Axiom (iii) for probability laws $\mathbf{P}(\emptyset) = 0$.

(c) If events A, B satisfy $\mathbf{P}(A) = .2$, $\mathbf{P}(B) = .3$ and $\mathbf{P}(A \cap B) = .1$, find $\mathbf{P}(A \cup B)$.

Using Proposition 2.33 from the notes, $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = .2 + .3 - .1 = .4$.

(d) In how many distinct ways can we arrange 6 objects from left to right?

From Proposition 2.67, this is the number of permutation of six elements, which is $6!$.

(e) If the events A, B are disjoint, what is $A \cap B$?

From the definition of disjointness, $A \cap B = \emptyset$.

2. Question 2

Roll a fair six-sided die and let N be the number that comes up. Let A be the event that N is even. Let B be the event that $N \leq 4$. Let C be the event that $N \leq 5$. Show that A and B are independent, but A and C are not independent.

Solution. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and let $\mathbf{P}$ be the uniform probability law on Ω. We have $A = \{2, 4, 6\}$, $B = \{1, 2, 3, 4\}$ and $C = \{1, 2, 3, 4, 5\}$. So, $A \cap B = \{2, 4\}$ and $A \cap C = \{2, 4\}$.

To show that A and B are independent, it suffices to show that $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$. By the definition of $\mathbf{P}$, we have $\mathbf{P}(A \cap B) = \mathbf{P}(\{2, 4\}) = 2/6 = 1/3 = (1/2)(4/6) = \mathbf{P}(A)\mathbf{P}(B)$.

To show that A and B are not independent, it suffices to show that $\mathbf{P}(A \cap C) \neq \mathbf{P}(A)\mathbf{P}(C)$. By the definition of $\mathbf{P}$, we have $\mathbf{P}(A \cap C) = \mathbf{P}(\{2, 4\}) = 2/6 \neq (1/2)(5/6) = \mathbf{P}(A)\mathbf{P}(C)$.

3. Question 3

There are n different cars labelled with numbers 1 to n. These cars are parked in n consecutive parking spaces from left to right in a uniformly random order. What is the probability that the cars labelled 1 and 2 are parked next to each other?

Solution. There are a couple of ways to do this problem. We will do this problem by counting all possible ways that the cars labelled 1 and 2 are parked next to each other.

To build up our intuition, we first consider some specific cases when n is small. When $n = 2$, this probability is 1, since the cars are always next to each other. When $n = 3$, there are $3!$ possible orderings: $123, 132, 231, 213, 312, 321$. By inspection, there are four orderings such that 1 is adjacent to 2. So, this event has probability $4/3! = 4/6 = 2/3$.

We now consider the general case. There are $n - 1$ positions of cars 1 and 2 such that car 1 is directly to the left of car 2. For example, if $*$ denotes any other car, the arrangements are $12 * * \cdots *$, $*12 * \cdots *$, $\ldots$, $* \cdots * 12*$ and $* \cdots * 12$. For each such position of cars 1 and 2, there are $(n - 2)!$ possible orderings of the remaining cars (using Proposition 2.67 from the notes).

[1]March 26, 2016,

Similarly, there are $n-1$ positions of cars 1 and 2 such that car 1 is directly to the right of car 2. For each such position of cars 1 and 2, there are $(n-2)!$ possible orderings of the remaining cars.

In summary, the total number of orderings of the cars such that 1 and 2 are adjacent is

$$(n-1)(n-2)! + (n-1)(n-2)! = 2(n-1)!.$$

Meanwhile, the total number of orderings of the cars is $n!$ (using Proposition 2.67 from the notes). Since all orderings are equally likely, the required probability is then

$$\frac{2(n-1)!}{n!} = \frac{2}{n}.$$

Note that this formula agrees with our explicit computations for $n = 2$ and $n = 3$.

4. Question 4

You have three boxes. The first one contains 1 white and 8 black balls, the second one contains 5 white and 4 black balls, and the last one contains 2 white and 1 black ball. You choose one of these three boxes uniformly at random, and then pick a ball from this box also uniformly at random. What is the probability you pick a white ball?

Solution. For any $i \in \{1,2,3\}$, let A_i be the event that the chosen box is the i^{th} box. Then $\cup_{i=1}^{3} A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for each $i \neq j$, $i, j \in \{1,2,3\}$. Note that $\mathbf{P}(A_i) = 1/3$ for each $i \in \{1,2,3\}$ by assumption. Let W be the event that the white ball is chosen. From the total probability theorem,

$$\mathbf{P}(W) = \sum_{i=1}^{3} \mathbf{P}(W|A_i)\mathbf{P}(A_i) = \frac{1}{3}\sum_{i=1}^{3} \mathbf{P}(W|A_i) = \frac{1}{3}(1/9 + 5/9 + 2/3) = 4/9.$$

5. Question 5

There are three museums in a town. Each is open with probability 0.4 and closed with probability 0.6 independently of the other museums. Two of the museums are located on the street A and one on the street B. A tourist chooses either street A or street B with equal probability and walks down the whole length of the street he chose. He will enter every open museum he encounters on his way. Given that he entered exactly one museum, what is the probability he chose to walk down the street A?

Solution. Let W be the event he walks down street A, and let M be the event that he entered exactly one museum. Using Bayes' rule (Theorem 2.50 in the notes),

$$\mathbf{P}(W|M) = \frac{\mathbf{P}(W)\mathbf{P}(M|W)}{\mathbf{P}(M|W)\mathbf{P}(W) + \mathbf{P}(M|W^c)\mathbf{P}(W^c)}.$$

If W is given, that is, if it is given that he walks down street A, then he enters both museums with probability $(.4)^2 = .16$ (since the probability that each museum is open is independent of the other), and he enters no museum with probability $(.6)^2 = .36$. So, he enters exactly one museum with probability $1 - .16 - .36 = .48$. In summary, $\mathbf{P}(M|W) = .48$.

If W^c is given, that is, if it is given that he walks down street B, then he enters the one museum on that street with probability $.4$. In summary, $\mathbf{P}(M|W^c) = .4$.

Lastly, it is given that either street is walked with equal probability, so $\mathbf{P}(W) = \mathbf{P}(W^c) = 1/2$. Plugging everything into our formula, we get

$$\mathbf{P}(W|M) = \frac{(1/2).48}{.48(1/2) + .4(1/2)} = \frac{.24}{.44} = \frac{6}{11}.$$