170B Midterm 2 Solutions[1]

1. Question 1

Label the following statements as TRUE or FALSE. If the statement is true, explain your reasoning. If the statement is false, provide a counterexample and explain your reasoning.

(a) Let X be a random variable. Let $i := \sqrt{-1}$. Then $\left|\mathbf{E}e^{itX}\right| \leq 1$ for any $t \in \mathbf{R}$.

TRUE. $\left|\mathbf{E}e^{itX}\right| \leq \mathbf{E}\left|e^{itX}\right| = 1$, using $|e^{itx}| = \sqrt{\cos^2(tx) + \sin^2(tx)} = 1$ for any $t, x \in \mathbf{R}$.

(b) Suppose I am flipping a coin over and over again. For any positive integer n, let A_n be the event that the n^{th} coin flip is heads. Suppose $\mathbf{P}(A_n) = n^{-2}$, for any positive integer n. Let B be the event that infinitely many of the coin flips are heads. Then $\mathbf{P}(B) = 0$.

TRUE. This follows from the Borel-Cantelli Lemma, since $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \sum_{n=1}^{\infty} n^{-2} < \infty$, so $\mathbf{P}(B) = 0$.

(c) Let $X_1, X_2, \ldots$ be independent random variables. Let $\mu := \mathbf{E}X_1$ and let $\sigma := \mathrm{var}(X_1)$. Assume $0 < \sigma < \infty$ and $-\infty < \mu < \infty$. Then, for any $t \in \mathbf{R}$,

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq t\right) = \int_{-\infty}^{t} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}.$$

FALSE. We made no mention of being identically distributed. To get a counterexample, let X_1 so that $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$ and let $X_n = 0$ for all $n \geq 2$. (Constant functions are automatically independent of other random variables.) Then $\mu = 0, \sigma = 1$ and $\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}} = \frac{X_1}{\sqrt{n}}$. So,

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq t\right) = \begin{cases} 0 & \text{if}\, t < 0 \\ 1 & \text{if}\, t \geq 0. \end{cases}$$

Moreover, the Central Limit Theorem should divide by the square root of the variance, not the variance, so this statement is false in two ways.

2. Question 2

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables such that $\mathbf{E}\,|X_1| < \infty$ and $\mathrm{var}(X_1) < \infty$. For any $n \geq 1$, define

$$Y_n := \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

Show that $Y_1, Y_2, \ldots$ converges in probability. Express the limit in terms of $\mathbf{E}X_1$ and $\mathrm{var}(X_1)$.

Solution. Note that $X_1^2, X_2^2, \ldots$ are independent, identically distributed random variables (since $X_1, X_2, \ldots$ are as well). (For example, $\mathbf{P}(X_i^2 \leq t) = \mathbf{P}(-\sqrt{t} \leq X_i \leq \sqrt{t}) = \mathbf{P}(-\sqrt{t} \leq X_1 \leq \sqrt{t}) = \mathbf{P}(X_1^2 \leq t)$ for any $t \geq 0$, $i \geq 1$, where the middle inequality follows since $X_1, X_2, \ldots$ are identically distributed.) So, the Weak Law of Large Numbers says that $Y_1, Y_2, \ldots$ converges in probability to its mean. In this case, we have for any $n \geq 1$,

$$\mathbf{E}Y_n = \frac{n}{n}\mathbf{E}X_1^2 = \mathbf{E}X_1^2 = \mathbf{E}X_1^2 - (\mathbf{E}X_1)^2 + (\mathbf{E}X_1)^2 = \mathrm{var}(X_1) + (\mathbf{E}X_1)^2.$$

So, $Y_1, Y_2, \ldots$ converges in probability to the constant random variable $\mathrm{var}(X_1) + (\mathbf{E}X_1)^2$.

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3. Question 3

Let X be a random variable. Let $X_1, X_2, \ldots$ be a sequence of random variables such that

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^4 = 0.$$

Prove that $X_1, X_2, \ldots$ converges in probability to X.

Solution. Let $\varepsilon > 0$. From Markov's Inequality,

$$0 \le \mathbf{P}(|X_n - X| > \varepsilon) = \mathbf{P}(|X_n - X|^4 > \varepsilon^4) \le \varepsilon^{-4}\mathbf{E}\,|X_n - X|^4.$$

So, letting $n \to \infty$ and using our assumption, we get $0 \le \lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) \le 0$. That is, $\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \varepsilon) = 0$. Since this holds for any $\varepsilon > 0$, we conclude that $X_1, X_2, \ldots$ converges in probability to X.

4. Question 4

Let $f, g\colon \mathbf{R} \to \mathbf{R}$. Recall that $(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$. Show that, for any $t \in \mathbf{R}$,

$$(f * g)(t) = (g * f)(t).$$

Solution.

$$\begin{aligned}
(f * g)(t) &= \int_{-\infty}^{\infty} f(x)g(t-x)dx, && \text{, by the definition of convolution} \\
&= -\int_{\infty}^{-\infty} f(t-u)g(u)du, && \text{, changing variables with } u = t - x \text{, so } du = -dx \\
&= \int_{-\infty}^{\infty} g(u)f(t-u)du, && \text{, by a property of integrals} \\
&= (g * f)(t), && \text{, by the definition of convolution}
\end{aligned}$$

5. Question 5

Let X, Y be independent random variables. Suppose X is uniformly distributed in $[0,1]$. And suppose Y has density given by

$$f_Y(t) = \begin{cases} 0 & \text{, if } t < 0 \\ t & \text{, if } 0 \le t \le 1 \\ 2 - t & \text{, if } 1 \le t \le 2 \\ 0 & \text{, if } t > 2. \end{cases}$$

Find the density of $X + Y$.

Solution 1. From Proposition 2.60 in the notes, $f_{X+Y} = f_X * f_Y$. So, we compute $f_X * f_Y$, by breaking into several cases. Using that $f_Y(t-x) = 0$ except when $0 \le t - x \le 2$, i.e. when $0 \ge x - t \ge -2$, i.e. when $t - 2 \le x \le t$, we have $f_X(x)f_Y(t-x) = 0$ except when

$0 \le x \le 1$ and $t-2 \le x \le t$.

$$\begin{aligned}
f_X * f_Y(t) &= \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)dx \qquad \text{, by the definition of convolution} \\
&= \int_{x=0}^{x=1} f_Y(t-x)dx \qquad \text{, by the definition of } f_X \\
&= \begin{cases} \int_{x=0}^{x=1} 0dx & \text{, if } t < 0 \\ \int_{x=0}^{x=t} f_Y(t-x)dx & \text{, if } 0 \le t \le 1 \\ \int_{x=0}^{x=1} f_Y(t-x)dx & \text{, if } 1 \le t \le 2 \\ \int_{x=t-2}^{x=1} f_Y(t-x)dx & \text{, if } 2 \le t \le 3 \\ 0 & \text{, if } t > 3. \end{cases}
\end{aligned}$$

So, using the definition of f_Y, we get

$$\begin{aligned}
f_X * f_Y(t) &= \begin{cases} \int_{x=0}^{x=1} 0dx & \text{, if } t < 0 \\ \int_{x=0}^{x=t} (t-x)dx & \text{, if } 0 \le t \le 1 \\ \int_{x=0}^{x=t-1} f_Y(t-x)dx + \int_{x=t-1}^{x=1} f_Y(t-x)dx & \text{, if } 1 \le t \le 2 \\ \int_{x=t-2}^{x=1} (2-(t-x))dx & \text{, if } 2 \le t \le 3 \\ 0 & \text{, if } t > 3. \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 0 \\ [tx - x^2/2]_{x=0}^{x=t} & \text{, if } 0 \le t \le 1 \\ \int_{x=0}^{x=t-1} (2-(t-x))dx + \int_{x=t-1}^{x=1} (t-x)dx & \text{, if } 1 \le t \le 2 \\ [(2-t)x + x^2/2]_{x=t-2}^{x=1} & \text{, if } 2 \le t \le 3 \\ 0 & \text{, if } t > 3. \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 0 \\ t^2/2 & \text{, if } 0 \le t \le 1 \\ [(2-t)x + x^2/2]_{x=0}^{x=t-1} + [tx - x^2/2]_{x=t-1}^{x=1} & \text{, if } 1 \le t \le 2 \\ (2-t+1/2+(2-t)^2-(2-t)^2/2) & \text{, if } 2 \le t \le 3 \\ 0 & \text{, if } t > 3. \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 0 \\ t^2/2 & \text{, if } 0 \le t \le 1 \\ (2-t)(t-1) + (t-1)^2/2 + t - 1/2 - t(t-1) + (t-1)^2/2 & \text{, if } 1 \le t \le 2 \\ t^2/2 - 3t + 9/2 & \text{, if } 2 \le t \le 3 \\ 0 & \text{, if } t > 3. \end{cases} \\
&= \begin{cases} 0 & \text{, if } t < 0 \\ t^2/2 & \text{, if } 0 \le t \le 1 \\ -t^2 + 3t - 3/2 & \text{, if } 1 \le t \le 2 \\ t^2/2 - 3t + 9/2 & \text{, if } 2 \le t \le 3 \\ 0 & \text{, if } t > 3. \end{cases}
\end{aligned}$$

Solution 2. From Proposition 2.60 in the notes, $f_{X+Y} = f_X * f_Y$. So, we compute $f_X * f_Y$, by breaking into several cases. From the previous problem, $f_X * f_Y = f_Y * f_X$, so we compute $f_Y * f_X$. Note that $f_X(t-x) = 1$ when $0 \leq t - x \leq 1$, and $f_X(t-x) = 0$ otherwise. So, $f_X(t-x) = 1$ when $0 \geq x - t \geq -1$, i.e. when $-1 + t \leq x \leq t$. These inequalities constrain the integral defining $f_Y * f_X(t)$ as follows.

$$\begin{aligned} f_Y * f_X(t) &= \int_{-\infty}^{\infty} f_Y(x) f_X(t-x) dx \qquad &&\text{, by the definition of convolution} \\ &= \int_{x=t-1}^{x=t} f_Y(x) dx \qquad &&\text{, by the definition of } f_X \end{aligned}$$

Substituting now the (piecewise) definition of f_Y,

$$f_Y * f_X(t) = \begin{cases} \int_{x=t-1}^{x=t} 0 dx & \text{, if } t < 0 \\ \int_{x=0}^{x=t} x dx & \text{, if } 0 \leq t \leq 1 \\ \int_{x=t-1}^{x=1} x dx + \int_{x=1}^{x=t} (2-x) dx & \text{, if } 1 \leq t \leq 2 \\ \int_{x=t-1}^{x=2} (2-x) dx & \text{, if } 2 \leq t \leq 3 \\ 0 & \text{, if } t > 3. \end{cases}$$

$$= \begin{cases} 0 & \text{, if } t < 0 \\ t^2/2 & \text{, if } 0 \leq t \leq 1 \\ (1 - (t-1)^2)/2 + 2(t-1) - (t^2-1)/2 & \text{, if } 1 \leq t \leq 2 \\ 2(2-(t-1)) - (4 - (t-1)^2)/2 & \text{, if } 2 \leq t \leq 3 \\ 0 & \text{, if } t > 3. \end{cases}$$

$$= \begin{cases} 0 & \text{, if } t < 0 \\ t^2/2 & \text{, if } 0 \leq t \leq 1 \\ -t^2 + 3t - 3/2 & \text{, if } 1 \leq t \leq 2 \\ t^2/2 - 3t + 9/2 & \text{, if } 2 \leq t \leq 3 \\ 0 & \text{, if } t > 3. \end{cases}$$