Please provide complete and well-written solutions to the following exercises.

Due November 30, in the discussion section.

Homework 7

Exercise 1. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbf{P}(X_1 = 1) = \mathbf{P}(X_1 = -1) = 1/2$. For any $n \geq 1$, define

$$S_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}}.$$

The Central Limit Theorem says that S_n converges in distribution to a standard Gaussian random variable. We show that S_n does not converge in probability to any random variable. The intuition here is that if S_n did converge in probability to a random variable Z, then when n is large, S_n is close to Z, $Y_n := \frac{\sqrt{2}S_{2n} - S_n}{\sqrt{2}-1}$ is close to Z, but S_n and Y_n are independent. And this cannot happen.

Proceed as follows. Assume that S_n converges in probability to Z.

- Let $\varepsilon > 0$. For n very large (depending on ε), we have $\mathbf{P}(|S_n - Z| > \varepsilon) < \varepsilon$ and $\mathbf{P}(|Y_n - Z| > \varepsilon) < \varepsilon$.
- Show that $\mathbf{P}(S_n > 0, Y_n > 0)$ is around $1/4$, using independence and the Central Limit Theorem.
- From the first item, show $\mathbf{P}(S_n > 0 | Z > \varepsilon) > 1 - \varepsilon$, $\mathbf{P}(Y_n > 0 | Z > \varepsilon) > 1 - \varepsilon$, so $\mathbf{P}(S_n > 0, Y_n > 0 | Z > \varepsilon) > 1 - 2\varepsilon$.
- Without loss of generality, for ε small, we have $\mathbf{P}(Z > \varepsilon) > 4/9$.
- By conditioning on $Z > \varepsilon$, show that $\mathbf{P}(S_n > 0, Y_n > 0)$ is at least $3/8$, when n is large.

Exercise 2. Let $X_1, X_2, \ldots$ be random variables that converge almost surely to a random variable X. That is,

$$\mathbf{P}(\lim_{n\to\infty} X_n = X) = 1.$$

Show that $X_1, X_2, \ldots$ converges in probability to X in the following way.

- For any $\varepsilon > 0$ and for any positive integer n, let

$$A_{n,\varepsilon} := \bigcup_{m=n}^{\infty} \{\omega \in \Omega \colon |X_m(\omega) - X(\omega)| > \varepsilon\}.$$

  Show that $A_{n,\varepsilon} \supseteq A_{n+1,\varepsilon} \supseteq A_{n+2,\varepsilon} \supseteq \cdots$.
- Show that $\mathbf{P}(\cap_{n=1}^{\infty} A_{n,\varepsilon}) = 0$.
- Using Continuity of the Probability Law, deduce that $\lim_{n\to\infty} \mathbf{P}(A_{n,\varepsilon}) = 0$.

Now, show that the converse is false. That is, find random variables $X_1, X_2, \ldots$ that converge in probability to X, but where $X_1, X_2, \ldots$ do not converge to X almost surely.

Exercise 3. Using the Central Limit Theorem, prove the Weak Law of Large Numbers.

Exercise 4. Let $m \geq 1$. Show by integral comparison of infinite series that

$$\sum_{j=m}^{\infty} \frac{1}{j^2} \leq \frac{10}{m}.$$

Exercise 5 (Renewal Theory)**.** Let $t_1, t_2, \ldots$ be positive, independent identically distributed random variables. Let $\mu \in \mathbf{R}$. Assume $\mathbf{E}t_1 = \mu$. For any positive integer j, we interpret t_j as the lifetime of the j^{th} lightbulb (before burning out, at which point it is replaced by the $(j+1)^{st}$ lightbulb). For any $n \geq 1$, let $T_n := t_1 + \cdots + t_n$ be the total lifetime of the first n lightbulbs. For any positive integer t, let $N_t := \min\{n \geq 1 \colon T_n \geq t\}$ be the number of lightbulbs that have been used up until time t. Show that N_t/t converges almost surely to $1/\mu$ as $t \to \infty$. (Hint: by definition of N_t, we have $T_{N_t - 1} < t \leq T_{N_t}$. Now divide the inequalities by N_t and apply the Strong Law.)

Exercise 6 (Playing Monopoly Forever)**.** Let $t_1, t_2, \ldots$ be independent random variables, all of which are uniform on $\{1, 2, 3, 4, 5, 6\}$. For any positive integer j, we think of t_j as the result of rolling a single fair six-sided die. For any $n \geq 1$, let $T_n = t_1 + \cdots + t_n$ be the total number of spaces that have been moved after the n^{th} roll. (We think of each roll as the amount of moves forward of a game piece on a very large Monopoly game board.) For any positive integer t, let $N_t := \min\{n \geq 1 \colon T_n \geq t\}$ be the number of rolls needed to get t spaces away from the start. Using Exercise 5, show that N_t/t converges almost surely to $2/7$ as $t \to \infty$.

Exercise 7 (Random Numbers are Normal)**.** Let X be a uniformly distributed random variable on $(0, 1)$. Let X_1 be the first digit in the decimal expansion of X. Let X_2 be the second digit in the decimal expansion of X. And so on.

- Show that the random variables $X_1, X_2, \ldots$ are uniform on $\{0, 1, 2, \ldots, 9\}$ and independent.
- Fix $m \in \{0, 1, 2, \ldots, 9\}$. Using the Strong Law of Large Numbers, show that with probability one, the fraction of appearances of the number m in the first n digits of X converges to $1/10$ as $n \to \infty$.

(Optional): Show that for any ordered finite set of digits of length k, the fraction of appearances of this set of digits in the first n digits of X converges to 10^{-k} as $n \to \infty$. (You already proved the case $k = 1$ above.) That is, a randomly chosen number in $(0, 1)$ is normal. On the other hand, if we just pick some number such that $\sqrt{2} - 1$, then it may not be easy to say whether or not that number is normal.

(As an optional exercise, try to explicitly write down a normal number. This may not be so easy to do, even though a random number in $(0, 1)$ satisfies this property!)

Exercise 8. Let $X_1, X_2, \ldots$ be random variables with mean zero and variance one. The Strong Law of Large Numbers says that $\frac{1}{n}(X_1 + \cdots + X_n)$ converges almost surely to zero. The Central Limit Theorem says that $\frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$ converges in distribution to a standard Gaussian random variable. But what happens if we divide by some other power of n? This Exercise gives a partial answer to this question.

Let $\varepsilon > 0$. Show that

$$\frac{X_1 + \cdots + X_n}{n^{1/2}(\log n)^{(1/2)+\varepsilon}}$$

converges to zero almost surely as $n \to \infty$. (Hint: Re-do the proof of the Strong Law of Large Numbers, but divide by $n^{1/2}(\log n)^{(1/2)+\varepsilon}$ instead of n.)

Exercise 9. Let A, B be events in a sample space. Let $C_1, \ldots, C_n$ be events such that $C_i \cap C_j = \emptyset$ for any $i, j \in \{1, \ldots, n\}$, and such that $\cup_{i=1}^n C_i = B$. Show:

$$\mathbf{P}(A|B) = \sum_{i=1}^{n} \mathbf{P}(A|B,\, C_i)\mathbf{P}(C_i|B).$$

(Hint: consider using the Total Probability Theorem and that $\mathbf{P}(\cdot|B)$ is a probability law.)