Please provide complete and well-written solutions to the following exercises.

Due January 19, in the discussion section.

Homework 1

Exercise 1. Using the De Moivre-Laplace Theorem, estimate the probability that 1000000 coin flips of fair coins will result in more than $501,000$ heads. (Some of the following integrals may be relevant: $\int_{-\infty}^{0} e^{-t^2/2}dt/\sqrt{2\pi} = 1/2$, $\int_{-\infty}^{1} e^{-t^2/2}dt/\sqrt{2\pi} \approx .8413$, $\int_{-\infty}^{2} e^{-t^2/2}dt/\sqrt{2\pi} \approx .9772$, $\int_{-\infty}^{3} e^{-t^2/2}dt/\sqrt{2\pi} \approx .9987$.)

Casinos do these kinds of calculations to make sure they make money and that they do not go bankrupt. Financial institutions and insurance companies do similar calculations for similar reasons.

Exercise 2. Let X and Y be nonnegative random variables. Recall that we can define

$$\mathbf{E}X := \int_0^\infty \mathbf{P}(X > t)dt.$$

Assume that $X \leq Y$. Conclude that $\mathbf{E}X \leq \mathbf{E}Y$.

More generally, if X satisfies $\mathbf{E}|X| < \infty$, we define $\mathbf{E}X := \mathbf{E}\max(X,0) - \mathbf{E}\max(-X,0)$. If X,Y are any random variables with $X \leq Y$, $\mathbf{E}|X| < \infty$ and $\mathbf{E}|Y| < \infty$, show that $\mathbf{E}X \leq \mathbf{E}Y$.

Exercise 3. Using the definition of convergence, show that the sequence of numbers

$$1, 1/2, 1/3, 1/4, \ldots$$

converges to 0.

Exercise 4 (Uniqueness of limits)**.** Let $x_1, x_2, \ldots$ be a sequence of real numbers. Let $x, y \in \mathbf{R}$. Assume that $x_1, x_2, \ldots$ converges to x. Assume also that $x_1, x_2, \ldots$ converges to y. Prove that $x = y$. That is, a sequence of real numbers cannot converge to two different real numbers.

Exercise 5. Let X be a uniformly distributed random variable on $[-1,1]$. Let $Y := X^2$. Find f_Y.

Exercise 6. Let X be a uniformly distributed random variable on $[0,1]$. Let $Y := 4X(1-X)$. Find f_Y.

Exercise 7. Let X be a uniformly distributed random variable on $[0,1]$. Find the PDF of $-\log(X)$.

Exercise 8. Let X be a standard normal random variable. Find the PDF of e^X.

Exercise 9. Let X,Y,Z be independent standard Gaussian random variables. Find the PDF of $\max(X,Y,Z)$.

Exercise 10. Let X be a random variable uniformly distributed in $[0, 1]$ and let Y be a random variable uniformly distributed in $[0, 2]$. Suppose X and Y are independent. Find the PDF of X/Y^2.