Please provide complete and well-written solutions to the following exercises.

Due November 23, at the beginning of class.

Homework 7

Exercise 1. Suppose you run a (busy) car wash, and the number of cars that come to the car wash between time 0 and time $s > 0$ is a Poisson poisson with rate $\lambda = 1$. Suppose each car is equally likely to have one, two, three, or four people in it. What is the average number of cars with four people that have arrived by time $s = 100$?

Exercise 2. Let X be a Poisson random variable with parameter $\lambda > 0$. Let Y be a Poisson random variable with parameter $\delta > 0$. Assume that X, Y are independent. Then $X + Y$ is a Poisson random variable with parameter $\lambda + \delta$.

Exercise 3. Suppose you are still running a (busy) car wash. The number of red cars that come to the car wash between time 0 and time $s > 0$ is a Poisson poisson with rate 2. The number of blue cars that come to car wash between time 0 and time $s > 0$ is a Poisson poisson with rate 3. Both Poisson processes are independent of each other. All cars are either red or blue. With what probability will five blue cars arrive, before three red cars have arrived?

Exercise 4. Prove the following two facts, which we used in the proof of the Law of Large Numbers for Renewal Processes.

Let $X_1, X_2, \ldots, Y_1, Y_2, \ldots, Z_1, Z_2, \ldots$ be random variables. Let $a, b \in \mathbf{R}$.

- Assume that $X_n \leq Y_n \leq Z_n$ for any $n \geq 1$. Assume that $\mathbf{P}(\lim_{n\to\infty} X_n = a) = 1$ and $\mathbf{P}(\lim_{n\to\infty} Z_n = a) = 1$. Prove that $\mathbf{P}(\lim_{n\to\infty} Y_n = a) = 1$.
- Assume that $\mathbf{P}(\lim_{n\to\infty} X_n = a) = 1$ and $\mathbf{P}(\lim_{n\to\infty} Y_n = b) = 1$. Prove that $\mathbf{P}(\lim_{n\to\infty} X_n Y_n = ab) = 1$.