Please provide complete and well-written solutions to the following exercises.

Due February 7, in the discussion section.

Homework 4

Exercise 1 (Knight Moves)**.** Consider a standard 8×8 chess board. Let V be a set of vertices corresponding to each square on the board (so V has 64 elements). Any two vertices $x, y \in V$ are connected by an edge if and only if a knight can move from x to y. (The knight chess piece moves in an L-shape, so that a single move constitutes two spaces moved along the horizontal axis followed by one move along the vertical axis (or two spaces moved along the vertical axis, followed by one move along the horizontal axis.) Consider the simple random walk on this graph. This Markov chain then represents a knight randomly moving around a chess board. For every space x on the chessboard, compute the expected return time $\mathbf{E}_x T_x$ for that space. (It might be convenient to just draw the expected values on the chessboard itself.)

Exercise 2. Give an example of a random walk on a finite graph that is not reversible.

Exercise 3. Let P be the transition matrix of a finite, irreducible, reversible Markov chain with state space Ω and stationary distribution π. Let $f, g \in \mathbf{R}^{|\Omega|}$ be column vectors. Consider the following bilinear function on f, g, which is referred to as an inner product (or dot product):

$$\langle f, g \rangle := \sum_{x \in \Omega} f(x) g(x) \pi(x).$$

Show that P is self-adjoint (i.e. symmetric) in the sense that

$$\langle f, Pg \rangle = \langle Pf, g \rangle.$$

In particular (for those that have taken 115A), the spectral theorem implies that all eigenvalues of P are real.

Finally, find a transition matrix P such that at least one eigenvalue of P is not real.

Exercise 4 (Ehrenfest Urn Model)**.** Suppose we have two urns and n spheres. Each sphere is in either of the first or the second urn. At each step of the Markov chain, one of the spheres is chosen uniformly and random and moved from its current urn to the other urn. Let X_n be the number of spheres in the first urn at time n. Then the transition matrix defining the Markov chain is

$$P(j,k) = \begin{cases} \frac{n-j}{n} & \text{, if } k = j+1 \\ \frac{j}{n} & \text{, if } k = j-1 \\ 0 & \text{, otherwise.} \end{cases}$$

Show that the unique stationary distribution for this Markov chain is a binomial with parameters n and $1/2$.

Exercise 5. Let $V = \{0,1\}^n$ be a set of vertices. We construct a graph from V as follows. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \{0,1\}^n$. Then x and y are connected by an edge in the graph if and only if $\sum_{i=1}^{n} |x_i - y_i| = 1$. That is, x and y are connected if and only if they differ by a single coordinate.

For any $x \in V$, define $f(x) = \sum_{i=1}^{n} x_i$, $f \colon V \to \{0, 1, \ldots, n\}$. Given $x \in V$, we identify x with the state in the Ehrenfest urn model where the first urn has exactly $f(x)$ spheres. Show that the Ehrenfest urn model is a **projection** of the simple random walk on V in the following sense. The probability that $x \in V$ transitions to any state $z \in V$ such that $y = f(z)$ is equal to: the probability that Ehrenfest model with state $f(x)$ transitions to state y.

Moreover, the unique stationary distribution for the simple random walk on V can be projected to give the unique stationary distribution in the Ehrenfest model. That is, if π is the unique stationary distribution for the simple random walk on V, and if for any $A \subseteq \{0, 1, \ldots, n\}$, we define $\mu(A) = \pi(f^{-1}(A))$, then μ is Binomial with parameters n and $1/2$. (Here $f^{-1}(A) = \{x \in V \colon f(x) \in A\}$.)

Exercise 6 (Birth-and-Death Chains)**.** A birth-and-death chain can model the size of some population of organisms. Fix a positive integer k. Consider the state space $\Omega = \{0, 1, 2, \ldots, k\}$. The current state is the current size of the population, and at each step the size can increase or decrease by at most 1. We define $\{(p_n, r_n, q_n)\}_{n=0}^{k}$ such that $p_n + r_n + q_n = 1$ for each n, and

- $P(n, n+1) = p_n > 0$ for every $0 \leq n < k$.
- $P(n, n-1) = q_n > 0$ for every $0 < n \leq k$.
- $P(n, n) = r_n \geq 0$ for every $0 \leq n \leq k$.
- $q_0 = p_k = 0$.

Show that the birth-and-death chain is reversible.