Please provide complete and well-written solutions to the following exercises.

Due March 7, at the beginning of discussion section.

Homework 7

Exercise 1. Prove the following variant of the Optional Stopping Theorem. Assume that $(M_0, M_1, \ldots)$ is a submartingale, and let T be a stopping time such that $\mathbf{P}(T < \infty)$. Let $c \in \mathbf{R}$. Assume that $|M_{n\wedge T}| \leq c$ for all $n \geq 0$. Then $\mathbf{E}M_T \geq \mathbf{E}M_0$. That is, you can make money by stopping a submartingale.

Exercise 2 (Ballot Theorem)**.** Let a, b be positive integers. Suppose there are c votes cast by c people in an election. Candidate 1 gets a votes and candidate 2 gets b votes. (So $c = a + b$.) Assume $a > b$. The votes are counted one by one. The votes are counted in a uniformly random ordering, and we would like to keep a running tally of who is currently winning. (News agencies seem to enjoy reporting about this number.) Suppose the first candidate eventually wins the election. We ask: with what probability will candidate 1 always be ahead in the running tally of who is currently winning the election? As we will see, the answer is $\frac{a-b}{a+b}$.

To prove this, for any positive integer k, let S_k be the number of votes for candidate 1, minus the number of votes for candidate 2, after k votes have been counted. Then, define $X_k := S_{c-k}/(c-k)$. Show that $X_0, X_1, \ldots$ is a martingale with respect to $S_c, S_{c-1}, S_{c-2}, \ldots$. Then, let T such that $T = \min\{0 \leq k \leq c \colon X_k = 0\}$, or $T = c - 1$ if no such k exists. Apply the Optional Stopping theorem to X_T to deduce the result.

Exercise 3. Let $(X_0, X_1, \ldots)$ be the simple random walk on $\mathbf{Z}$. For any $n \geq 0$, define $M_n = X_n^3 - 3nX_n$. Show that $(M_0, M_1, \ldots)$ is a martingale with respect to $(X_0, X_1, \ldots)$

Now, fix $m > 0$ and let T be the first time that the walk hits either 0 or m. Show that, for any $0 < k \leq m$,

$$\mathbf{E}_k(T \mid X_T = m) = \frac{m^2 - k^2}{3}.$$

(If you use the Optional Stopping Theorem, you do not have to verify that the martingale is bounded.)

Exercise 4. Let $X_1, X_2, \ldots$ be independent random variables with $\mathbf{E}X_i = 0$ for every $i \geq 1$. Suppose there exists $\sigma > 0$ such that $\mathrm{Var}(X_i) = \sigma^2$ for all $i \geq 1$. For any $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Show that $S_n^2 - n\sigma^2$ is a martingale with respect to $X_1, X_2, \ldots$. (We let $X_0 = 0$.)

Let $a > 0$. Let $T = \min\{n \geq 1 \colon |S_n| \geq a\}$. Using the Optional Stopping Theorem, show that $\mathbf{E}T \geq a^2/\sigma^2$. Observe that a simple random walk on $\mathbf{Z}$ has $\sigma^2 = 1$ and $\mathbf{E}T = a^2$ when $a \in \mathbf{Z}$.

(If you use the Optional Stopping Theorem, you do not have to verify that the martingale is bounded.)

Exercise 5. Let $\lambda > 0$. Let $\tau_1, \tau_2, \ldots$ be independent exponential random variables with parameter λ. For any $n \geq 1$, let $T_n = \tau_1 + \cdots + \tau_n$. Fix positive integers $n_k > \cdots > n_1$ and positive real numbers $t_k > \cdots > t_1$. Then

$$f_{T_{n_k},\ldots,T_{n_1}}(t_k, \ldots, t_1) = f_{T_{(n_k - n_{k-1})}}(t_k - t_{k-1}) \cdots f_{T_{(n_2 - n_1)}}(t_2 - t_1) f_{T_{n_1}}(t_1).$$

(Hint: just try to case $k = 2$ first, and use a conditional density function.)

Exercise 6. Let $s, t > 0$ and let m, n be nonnegative integers. Let $0 < t_m < t_{m+1} < t_{m+n} < t_{m+n+1}$, and define (using the notation of Exercise 5),

$$g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) := f_{T_1}(t_{m+n+1} - t_{m+n}) f_{T_{n-1}}(t_{m+n} - t_{m+1}) f_{T_1}(t_{m+1} - t_m) f_{T_m}(t_m).$$

Let $\{N(s)\}_{s \geq 0}$ be a Poisson Process with parameter $\lambda > 0$. Show that

$$\begin{aligned}&\mathbf{P}(N(s+t) = m+n,\, N(s) = m)\\ &= \int_0^s \left(\int_s^{s+t} \left(\int_{t_{m+1}}^{s+t} \left(\int_{s+t}^{\infty} g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) dt_{m+n+1} \right) dt_{m+n} \right) dt_{m+1} \right) dt_m.\end{aligned}$$

(Hint: use the joint density, and then use Exercise 5.)

Exercise 7. Let $Y_1, Y_2, \ldots$ be independent identically distributed random variables. Let N be an independent, nonnegative integer-valued random variable. Let $S = Y_1 + \cdots + Y_N$, where $S := 0$ if $N = 0$.

- If $\mathbf{E}\,|Y_1| < \infty$ and $\mathbf{E}N < \infty$, then $\mathbf{E}S = (\mathbf{E}N)(\mathbf{E}Y_1)$.
- If $\mathbf{E}Y_1^2 < \infty$ and $\mathbf{E}N^2 < \infty$, then $\mathrm{var}(S) = (\mathbf{E}N)(\mathrm{var}(Y_1)) + (\mathbf{E}Y_1)^2(\mathrm{var}(N))$.
- If N is a Poisson random variable with parameter $\lambda > 0$, then $\mathrm{var}(S) = \lambda \mathbf{E}Y_1^2$.

(Hint: for the second part, use $\mathbf{E}(S^2 | N = n) = n \cdot \mathrm{var}(Y_1) + (n\mathbf{E}Y_1)^2$. Use this to compute $\mathbf{E}S^2$. Then compute $\mathrm{var}(S)$.)

Exercise 8. Suppose the number of students going to a restaurant in Ackerman in a single day has a Poisson distribution with mean 500. Suppose each student spends an average of \$10 with a standard deviation of \$5. What is the average revenue of the restaurant in one day? What is the standard deviation of the revenue in one day? (The amounts spent by the students are independent identically distributed random variables.)

Exercise 9. Give an example of a right-continuous function. Then give an example of a function that is not right-continuous.