Math 174E, Spring 2017, UCLA — Instructor: Steven Heilman

Name: ______________________ UCLA ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 1

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam. You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	9	
2	10	
3	10	
4	10	
5	10	
Total:	49	

Do not write in the table to the right. Good luck![a]

Reference sheet

Below are some definitions that may be relevant.

Let $(X_0, X_1, \ldots)$ be a real-valued stochastic process. A **real-valued martingale with respect to** $(X_0, X_1, \ldots)$ is a stochastic process $(M_0, M_1, \ldots)$ such that $\mathbf{E}\,|M_n| < \infty$ for all $n \geq 0$, and for any $m_0, x_0, \ldots, x_n \in \mathbf{R}$,

$$\mathbf{E}(M_{n+1} - M_n | X_n = x_n, \ldots, X_0 = x_0, M_0 = m_0) = 0.$$

A **stopping time** for a martingale $M_0, M_1, \ldots$ is a random variable T taking values in $0, 1, 2, \ldots, \cup\{\infty\}$ such that, for any integer $n \geq 0$, the event $\{T = n\}$ is determined by $M_0, X_0, \ldots, X_n$. More formally, for any integer $n \geq 1$, there is a set $B_n \subseteq \Omega^{n+2}$ such that $\{T = n\} = \{(M_0, X_0, \ldots, X_n) \in B_n\}$. Put another way, the indicator function $1_{\{T=n\}}$ is a function of the random variables $M_0, X_0, \ldots, X_n$.

Let X be a random variables on a sample space Ω. Let $A \subseteq \Omega$ with $\mathbf{P}(A) > 0$. Then the **conditional expectation of X given** A, denoted $\mathbf{E}(X|A)$ is

$$\mathbf{E}(X|A) := \frac{\mathbf{E}(X \cdot 1_A)}{\mathbf{P}(A)}.$$

Suppose we have a partition of a sample space Ω. That is, we have sets $A_1, \ldots, A_k \subseteq \Omega$ such that $A_i \cap A_j = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$, and $\cup_{i=1}^{k} A_i = \Omega$. Denote $\mathcal{A} = \{A_1, \ldots, A_k\}$. Define $\mathbf{E}(X|\mathcal{A})$ to be a random variable that takes the value $\mathbf{E}(X|A_i)$ on the set A_i.

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (3 points) For any positive integers i, j, let a_{ij} be a real number. Then

   $$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right).$$

   TRUE FALSE (circle one)

   (b) (3 points) Let $M_0 = 0$ and let $M_0, M_1, \ldots$ be a martingale. Let T be a stopping time for the martingale. Then $\mathbf{E}M_T = 0$.

   TRUE FALSE (circle one)

   (c) (3 points) Let X, Y be discrete random variables such that

   $$\mathbf{P}(X \leq x, Y = y) = \mathbf{P}(X \leq x)\mathbf{P}(Y = y), \qquad \forall\, x, y \in \mathbf{R}.$$

   Then

   $$\mathbf{P}(X \leq x, Y \leq y) = \mathbf{P}(X \leq x)\mathbf{P}(Y \leq y), \qquad \forall\, x, y \in \mathbf{R}.$$

   TRUE FALSE (circle one)

2. (10 points) For any $x \in \mathbf{R}$, define

$$\phi(x) := \max\Big(-x-1,\, 0,\, x-1\Big).$$

Prove that $\phi\colon \mathbf{R} \to \mathbf{R}$ is convex.

(In this problem, unlike the other problems, you are allowed to use results from the homework.)

3. (10 points) Suppose you begin at the lower left corner of a 4×4 chess board. Every day, you are allowed to move either up or right to a consecutive board space (unless you are waiting). When you land on a new space, you have to wait a number of days specified by the number sitting on that board space, until you move again. The numbers on the board spaces appear below.

$$\begin{pmatrix} 3 & 3 & 7 & 0 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 4 & 1 \\ 0 & 2 & 7 & 5 \end{pmatrix}.$$

Your goal is to reach the top right corner of the chess board in the shortest amount of time. Find the path that takes the shortest amount of time, and also find the shortest amount of time that it takes to reach the top right corner.

4. (10 points) Let $X \geq 0$ be a random variable such that $\mathbf{P}(X > 0) > 0$. Show that

$$\mathbf{E}(X \mid X > 0) \leq \frac{\mathbf{E}X^2}{\mathbf{E}X}.$$

(Hint: you can freely use the Cauchy-Schwarz inequality: $(\mathbf{E}XY)^2 \leq \mathbf{E}X^2\mathbf{E}Y^2$.)

5. (10 points) Let $X_0 = 0$, and let $a < 0 < b$ be integers. Let $0 < p < 1$ with $p \neq 1/2$. Let $X_1, X_2, \ldots$ be independent identically distributed random variables so that $\mathbf{P}(X_i = 1) = p$ and $\mathbf{P}(X_i = -1) = 1 - p$ for all $i \geq 1$. For any $n \geq 0$, let $Y_n := X_0 + \cdots + X_n$. Define $T := \min\{n \geq 1 \colon Y_n \notin (a, b)\}$.

   Compute $\mathbf{E}T$, in terms of a, b, p.

   (Hint: use martingales, somehow. If you use the Optional Stopping Theorem, you do not have to verify that the martingale is bounded.)

   (Second hint: you can freely use the formula $\mathbf{P}(Y_T = a) = \frac{(q/p)^{x_0} - (q/p)^b}{(q/p)^a - (q/p)^b}$, where $q := 1 - p$.)

(Scratch paper)