174E Midterm 2 Solutions, Spring 2017[1]

1. Question 1

Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Let $t > s > 0$. Show that $\mathbb{E}B(s)B(t) = s$.
- Let $a, b > 0$. Let $T_a := \min\{t \geq 0\colon B(t) = a\}$. Show that
$$\mathbf{P}(T_a < T_{-b}) = \frac{b}{a+b}$$
(If you use the Optional Stopping Theorem, then the only assumption you need to verify is that you have a martingale.)

Using that $B(s)$ has variance s, and using the independent increment property,

$$\begin{aligned}\mathbb{E}B(s)B(t) &= \mathbb{E}B(s)(B(t) - B(s) + B(s)) = \mathbb{E}(B(s))^2 + \mathbb{E}[B(s)(B(t) - B(s))]\\ &= s + [\mathbb{E}B(s)][\mathbb{E}(B(t) - B(s))] = s.\end{aligned}$$

Let $c := \mathbf{P}(T_a < T_{-b})$. Let $T := \min\{t \geq 0\colon B(t) \in \{a, -b\}\}$. From the Optional Stopping Theorem (for continuous-time martingales) (noting that $|B(t \wedge T)| \leq \max(a, b)$ for all $t \geq 0$)

$$0 = \mathbb{E}B(0) = \mathbb{E}B(T) = ac - b(1 - c).$$

Solving for c proves the result.

2. Question 2

Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion.

- Show that $\{(B(t))^2 - t\}_{t\geq 0}$ is a (continuous-time) martingale.
- Let $a, b > 0$. Let $T = \min\{t \geq 0\colon B(t) \notin (-b, a)\}$. Show that
$$\mathbb{E}T = ab.$$
(If you use the Optional Stopping Theorem, then the only assumption you need to verify is that you have a martingale.)

Solution. Using property (i) and then property (ii) of Brownian motion,

$$\begin{aligned}&\mathbb{E}((B(t))^2 - t - ((B(s))^2 - s) \,|\, B(s_n) = x_n, \ldots, B(s_1) = x_1)\\ &= \mathbb{E}((B(t) - B(s) + B(s) - B(s_n) + x_n)^2 - t - ((B(s) - B(s_n) + x_n)^2 - s)\\ &\qquad\qquad |\, B(s_n) = x_n, \ldots, B(s_1) = x_1)\\ &= \mathbb{E}(B(t) - B(s))^2 + \mathbb{E}(B(s) - x_n)^2 + x_n^2 - t - \mathbb{E}(B(s) - x_n)^2 - x_n^2 + s\\ &= (t - s) - t + s = 0.\end{aligned}$$

So, using the Optional stopping theorem, we get $0 = \mathbb{E}((B(T))^2 - T)$, then using the previous problem,

$$\begin{aligned}\mathbb{E}T &= \mathbb{E}(B(T))^2 = a^2\mathbf{P}(B(T) = a) + b^2\mathbf{P}(B(T) = -b)\\ &= a^2\frac{b}{a+b} + b^2\left(1 - \frac{b}{a+b}\right) = a^2\frac{b}{a+b} + b^2\frac{a}{a+b} = ab\frac{a+b}{a+b} = ab.\end{aligned}$$

3. Question 3

Consider a European call option and a European put option on a nondividend-paying stock. The following things are given

- The current price of the stock is 60.
- The call option currently sells for 0.15 more than the put option.
- Both the call option and put option will expire in 4 years.
- Both the call option and put option have a strike price of 70.

Calculate the continuously compounded risk-free interest rate. (That is, compute the interest rate r that ensures that no arbitrage opportunity exists.)

Solution. Put-call parity says that $S_0+p-c=ke^{-rt}$. It is given that $c-p=.15$, $S_0=60$, $k=70$ and $t=4$. So, $e^{-4r}=(60-.15)/70$, so $r=-(1/4)\log((60-.15)/70)\approx .0392$.

4. Question 4

The value of a lookback call option with strike price $k>0$ at time $t>0$ is

$$e^{-(\mu+\sigma^2/2)t}\mathbb{E}\max\left(\max_{0\leq r\leq t}S(r)-k,0\right).$$

Here $\{S(t)\}_{t\geq 0}=\{S_0e^{\sigma B(t)+\mu t}\}_{t\geq 0}$ is a geometric Brownian motion.

Write down an explicit integral formula for $\mathbb{E}\max\left(\max_{0\leq r\leq t}S(r)-k,0\right)$.

Solution. From an exercise from the notes, $\mathbf{P}(\max_{0\leq s\leq t}B(s)\geq x)=1-\int_{-x}^{x}e^{-\frac{y^2}{2t}}\frac{dy}{\sqrt{2\pi t}}$. So, by the Fundamental Theorem of calculus, $Z:=\max_{0\leq s\leq t}B(s)$ has density

$$f_Z(x)=2e^{-x^2/(2t)}\frac{1}{\sqrt{2\pi t}},\qquad x>0$$

We first assume that $\mu=0$. Since $\max_{0\leq s\leq t}e^{\sigma B(s)}=e^{\sigma\max_{0\leq s\leq t}B(s)}$,

$$\mathbb{E}\max\left(\max_{0\leq r\leq t}S(r)-k,0\right)=2\int_0^\infty\max(S_0e^{\sigma x}-k,0)e^{-x^2/(2t)}\frac{1}{\sqrt{2\pi t}}dx.$$

In the case $\mu\neq 0$, this question was unintentionally difficult. More generally, if $Y:=\max_{0\leq s\leq t}(\sigma B(s)+\mu s)$, then for any $y\geq 0$,

$$\mathbf{P}(Y\leq y)=\Phi\Big(\frac{y-\mu t}{\sigma\sqrt{t}}\Big)-e^{2\mu y/\sigma^2}\Phi\Big(\frac{-y-\mu t}{\sigma\sqrt{t}}\Big).$$

We can then differentiate this CDF to get the density and put it into the expected value formula, as above.

5. Question 5

In this problem, we consider the price of the **knockout option** using the binomial model.

Recall the binomial model. Let $u,d>0$. Let $0<p<1$. Let $(X_1,X_2,\ldots)$ be independent random variables such that $\mathbf{P}(X_n=\log u)=:p$ and $\mathbf{P}(X_n=\log d)=1-p\ \forall\ n\geq 1$. Let X_0 be a fixed constant. Let $Y_n:=X_0+\cdots+X_n$, and let $S_n:=e^{Y_n}\ \forall\ n\geq 1$. Let $r:=p(u-d)-1+d$. For any $n\geq 1$, define $M_n:=(1+r)^{-n}S_n$. Recall that $M_0,M_1,\ldots$ is a martingale, and S_n is the price of the stock at time n.

You are required to compute the price of a call option with strike price 28 and a knockout barrier of 20. We assume that $S_0=24$, $u=3/2$, $d=2/3$, $r=1/6$, so that $p=3/5$. The option expires at time $n = 3$, and you have the ability to exercise the option only at time $n = 3$. At any time $1 \leq n \leq 3$, if the price of the stock has dropped below the barrier value of 20, then the option becomes worthless at the current time and at any future time. Otherwise, if the stock has a current price s, then the option has a payoff of $\max(s - 28, 0)$.

Solution. We emulate the pricing of the American put option under the binomial model.

Let $S_{n,m}$ be the m^{th} value of S_n at time $0 \leq n \leq 3$. When $n = 3$, S_n can take four possible values: $S_0u^3, S_0u^2d, S_0u^1d^2$ and S_0d^3. That is, $S_{3,1} = 16(4/9)$, $S_{3,2} = 16$, $S_{3,3} = 36$ and $S_{3,4} = 81$.

Let $V_{n,m}$ be the value of the option at time n, when S_n takes the value $S_{n,m}$. If $S_{n,m}$ ever takes a value below 20, then $V_{n',m}$ is zero for any $n' \geq n$, but otherwise

$$\begin{aligned} V_{n,m} &= \frac{1}{1+r}(pV_{n,m+1} + (1-p)V_{n,m}) = \frac{6}{7}((3/5)V_{n+1,m+1} + (2/5)V_{n+1,m}) \\ &= \frac{6}{35}(3V_{n+1,m+1} + 2V_{n+1,m}), \qquad \forall\, 1 \leq m \leq n+1. \end{aligned}$$

We solve this recursion by slowly filling out a tree of values, as follows. Each entry in the matrix is $(S_{n,m}, V_{n,m})$.

$$\begin{pmatrix} & & & (81,53) \\ & & ? & (36,8) \\ & ? & ? & (16,0) \\ ? & ? & ? & (16(4/9),0) \end{pmatrix}, \qquad \begin{pmatrix} & & & (81,53) \\ & & (54,30) & (36,8) \\ & ? & (24,144/35) & (16,0) \\ ? & ? & (16(2/3),0) & (16(4/9),0) \end{pmatrix}$$

$$\begin{pmatrix} & & & (81,53) \\ & & (54,30) & (36,8) \\ & (36,3351/199) & (24,144/35) & (16,0) \\ ? & (16,0) & (16(2/3),0) & (16(4/9),0) \end{pmatrix}$$

So, the value of the option is $\frac{6}{35}(3\frac{3351}{199} + 2(0)) = \frac{2217}{256} \approx 8.660$.