Please provide complete and well-written solutions to the following exercises.

Due May 16, in the discussion section.

Homework 5

Exercise 1. Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and negative drift $\mu < 0$. Let $a < 0 < b$. Let $T := \min\{t \geq 0 \colon X(t) \in \{a, b\}\}$. Let $\alpha := 2|\mu|/\sigma^2$. Show that

$$\mathbf{E}T = \frac{1}{\mu} \cdot \frac{b(1 - e^{\alpha a}) + a(e^{\alpha b} - 1)}{e^{\alpha b} - e^{\alpha a}}$$

(If you use a martingale, you do not have to verify that it is bounded.)

Exercise 2. Let $\{X(t)\}_{t\geq 0} = \{\sigma B(t) + \mu t\}_{t\geq 0}$ be a standard Brownian motion with variance $\sigma^2 > 0$ and negative drift $\mu < 0$. Let $a < 0$. Let $T_a := \min\{t \geq 0 \colon X(t) = a\}$. Let $\alpha := 2|\mu|/\sigma^2$. Show that

$$\mathbf{E}T_a = \frac{a}{\mu}.$$

(If you use a martingale, you do not have to verify that it is bounded.)

Exercise 3 (Optional)**.** Write a computer program to simulate standard Brownian motion. More specifically, the program should simulate a random walk on $\mathbf{Z}$ with some small step size such as .002. (That is, simulate $B_k(t)$ when $k = 500^2$ and, say, $0 \leq t \leq 1$.)

Exercise 4 (Optional)**.** The following exercise assumes familiarity with Matlab and is derived from Cleve Moler's book, Numerical Computing with Matlab.

The file `brownian.m` plots the evolution of a cloud of particles that starts at the origin and diffuses in a two-dimensional random walk, modeling the Brownian motion of gas molecules.

(a) Modify `brownian.m` to keep track of both the average and the maximum particle distance from the origin. Using loglog axes, plot both sets of distances as functions of n, the number of steps. You should observe that, on the log-log scale, both plots are nearly linear. Fit both sets of distances with functions of the form $cn^{1/2}$. Plot the observed distances and the fits, using linear axes.

(b) Modify `brownian.m` to model a random walk in three dimensions. Do the distances behave like $n^{1/2}$?

The program `brownian.m` appears below.

```
% BROWNIAN   Two-dimensional random walk.
%   What is the expansion rate of the cloud of particles?
```

```
shg
clf
set(gcf,'doublebuffer','on')
delta = .002;
x = zeros(100,2);
h = plot(x(:,1),x(:,2),'.');
axis([-1 1 -1 1])
axis square
stop = uicontrol('style','toggle','string','stop');
while get(stop,'value') == 0
   x = x + delta*randn(size(x));
   set(h,'xdata',x(:,1),'ydata',x(:,2))
   drawnow
end
set(stop,'string','close','value',0,'callback','close(gcf)')
```

Exercise 5. Let $\mu \in \mathbf{R}$ and let $\sigma > 0$. Let X be a Gaussian random variable with mean μ and variance σ^2. Let $Y := e^X$. We then say Y has a **lognormal distribution with parameters** μ and σ^2. Show that Y has density

$$f(y) := \begin{cases} \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(\log(y)-\mu)^2}{2\sigma^2}} & \text{, if } y > 0 \\ 0 & \text{, if } y \leq 0. \end{cases}$$

Then, show that

$$\mathbf{E}Y = e^{\mu+\sigma^2/2}.$$
$$\mathbf{E}Y^2 = e^{2\mu+2\sigma^2}.$$

Exercise 6. In the context of Put-Call parity, show that an arbitrage opportunity exists if $S_0 + p - c > ke^{-rt}$. (That is, fill in the omitted details from the notes in this case.)

Exercise 7 (MFE Sample Question)**.** Consider a European call option and a European put option on a nondividend-paying stock. The following things are given

- The current price of the stock is 60.
- The call option currently sells for 0.15 more than the put option.
- Both the call option and put option will expire in 4 years.
- Both the call option and put option have a strike price of 70.

Calculate the continuously compounded risk-free interest rate. (That is, compute the interest rate r that ensures that no arbitrage opportunity exists.)

Exercise 8 (MFE Sample Question)**.** Near market closing time on a given day, you lose access to stock prices, but some European call and put prices for a stock are available as follows:

Strike Price	Call Price	Put Price
$40	$11	$3
$50	$6	$8
$55	$3	$11

All six options have the same expiration date.

After reviewing the information above, John tells Mary and Peter that no arbitrage opportunities can arise from these prices.

Mary disagrees with John. She argues that one could use the following portfolio to obtain arbitrage profit: Long one call option with strike price 40; short three call options with strike price 50; lend \$1; and long some calls with strike price 55. Peter also disagrees with John. He claims that the following portfolio, which is different from Marys, can produce arbitrage profit: Long 2 calls and short 2 puts with strike price 55; long 1 call and short 1 put with strike price 40; lend \$2; and short some calls and long the same number of puts with strike price 50.

Which of the following statements is true?

(A) Only John is correct.
(B) Only Mary is correct.
(C) Only Peter is correct.
(D) Both Mary and Peter are correct.
(E) None of them is correct.