307 Midterm 2 Solutions[1]

1. Question 1

(a) There is a random variable X such that

$$\mathrm{Var}(X) = -1.$$

FALSE. Variance is always nonnegative since $\mathrm{Var}(X) = \mathbf{E}(X - \mathbf{E}X)^2 \geq 0$. (b) Let X, Y be random variables. Then

$$\mathrm{cov}(X, Y) \geq 0.$$

FALSE. $\mathrm{cov}(X, -X) = -\mathrm{cov}(X, X) = -\mathrm{Var}(X)$, so an X with nonzero variance has $\mathrm{cov}(X, -X) < 0$.

(c) Let X, Y be random variables. Then

$$\mathbf{E}(XY) = (\mathbf{E}X)(\mathbf{E}Y).$$

FALSE. If X is a Bernoulli random variable and $Y = 1 - X$ then $XY = 0$ but $\mathbf{E}X > 0$ and $\mathbf{E}Y > 0$.

(d) Let X be a random variable and let $t > 0$ be a real number. Then

$$\mathrm{Var}(tX) = t^2\mathrm{Var}(X).$$

TRUE. $\mathrm{Var}(tX) = \mathbf{E}(tX)^2 - (\mathbf{E}tX)^2 = t^2[\mathbf{E}X^2 - (\mathbf{E}X)^2] = t^2\mathrm{Var}(X)$

(e) Let X, Y be random variables. Then

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

FALSE. $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$. So, if this covariance is nonzero, then the variance of the sum is not equal to the sum of variances.

2. Question 2

Suppose random variables X and Y have joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} xy & \text{, if } \ 0 \leq x \leq \sqrt{2} \text{ and } 0 \leq y \leq \sqrt{2}. \\ 0 & \text{, otherwise.} \end{cases}$$

- Compute the X marginal f_X. Simplify your answer as best you can.
- Compute the Y marginal f_Y. Simplify your answer as best you can.
- Are X and Y independent? Justify your answer.
- Compute $\mathbf{E}(XY + 1)$. Simplify your answer to the best of your ability.

Solution. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy = \int_0^{\sqrt{2}} xydy = x\int_0^{\sqrt{2}} ydy = x$ if $0 \leq x \leq 2$ and $f_X(x) = 0$ otherwise. $f_Y(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx = \int_0^{\sqrt{2}} xydx = y\int_0^{\sqrt{2}} xdx = y$ if $0 \leq y \leq 2$ and $f_Y(y) = 0$ otherwise. Since $f_{X,Y} = f_X f_Y$, X and Y are independent. Finally, by independence we have $\mathbf{E}(XY + 1) = \mathbf{E}X\mathbf{E}Y + 1$, and $\mathbf{E}X = \int_0^{\sqrt{2}} x f_X(x)dx = \int_0^{\sqrt{2}} x^2dx = 2\sqrt{2}/3$, And similarly $\mathbf{E}Y = 2\sqrt{2}/3$, so $\mathbf{E}(XY + 1) = 1 + (\sqrt{8}/3)^2 = 1 + 8/9 = 17/9$.

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3. Question 3

Suppose I flip a fair coin many times over and over again. (A fair coin has probability 1/2 of landing heads, and probability 1/2 of landing tails.)

Let X be the number of tails that occur before the first head appears.

Let Y be the number of tails that occur after the first head appears, and before the second head appears.

- Give a formula for the probability mass function p_X of X.
  Simplify to the best of your ability. Justify your answer.
- Give a formula for the probability mass function p_Y of Y.
  Simplify to the best of your ability. Justify your answer.
- Give a formula for the joint probability mass function $p_{X,Y}$ of X and Y.
  Simplify to the best of your ability. Justify your answer.

Solution. $X+1$ has a geometric distribution, so that $\mathbf{P}(X=k)=(1/2)^{k+1}$ for all $k\geq 0$. $Y+1$ also has a geometric distribution, so that $\mathbf{P}(Y=k)=(1/2)^{k+1}$ for all $k\geq 0$. By construction, X and Y are independent, so that $\mathbf{P}(X=j,Y=k)=\mathbf{P}(X=j)\mathbf{P}(Y=k)=(1/2)^{j+1}(1/2)^{k+1}$ for all $j,k\geq 0$.

4. Question 4

Suppose I roll a fair six-sided die n times. (A fair six-sided die has probability 1/6 of each of its six faces appearing after each roll.)

Let X be the number of times that the die face labelled "1" appears.

Let Y be the number of times that the die face labelled "2" appears.

- Compute $\mathrm{Var}(X)$. Simplify as best you can. Justify your answer.
- Compute $\mathrm{Cov}(X,Y)$. Simplify as best you can. Justify your answer.

Solution. For each $1\leq i\leq n$, let X_i be 1 if the die roll "1" occurs, and let $X_i=0$ otherwise. Then $X=X_1+\cdots+X_n$, and $X_1,\ldots,X_n$ are independent, so

$$\mathrm{Var}(X)=\mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_n).$$

We have $\mathbf{E}X_i=1/6$, and $\mathbf{E}X_i^2=1/6$, so $\mathrm{Var}(X_i)=1/6-(1/6)^2=5/36$ for all $1\leq i\leq n$. So,

$$\mathrm{Var}(X)=5n/36.$$

For each $1\leq i\leq n$, let Y_i be 1 if the die roll "2" occurs, and let $Y_i=0$ otherwise. Then $Y=Y_1+\cdots+Y_n$, and $Y_1,\ldots,Y_n$ are independent. Also

$$\mathrm{Cov}(X,Y)=\sum_{1\leq i,j\leq n}\mathrm{Cov}(X_i,Y_j).$$

If $i\neq j$, X_i and Y_j are independent, so that $\mathrm{Cov}(X_i,Y_j)=0$. So,

$$\mathrm{Cov}(X,Y)=\sum_{i=1}^{n}\mathrm{Cov}(X_i,Y_i).$$

For any $1\leq i\leq n$, we have $\mathbf{E}X_iY_i=0$, since $X_iY_i=0$ (it cannot occur that the die roll is both 1 and 2 simultaneously). So, $\mathrm{Cov}(X_i,Y_i)=\mathbf{E}X_iY_i-\mathbf{E}X_i\mathbf{E}Y_i=-\mathbf{E}X_i\mathbf{E}Y_i=-(1/6)(1/6)=-1/36$. In summary,

$$\mathrm{Cov}(X,Y)=-n/36.$$

5. Question 5

Suppose you have a standard 52 card deck of playing cards. The deck is shuffled, so that each of the 52! arrangements of the cards are equally likely to occur.

Compute the expected number of cards you have to draw from the top of the deck before you see a heart appear.

(For example, if the top card on the deck is itself a heart, you had to draw zero cards before you see a heart appear.)

(A standard 52 card deck has 13 hearts and 39 non-heart cards.)

Solution. Label each non-heart card with the an integer $1 \leq i \leq 39$. Let X_i be 1 if the i^{th} card appears above every heart in the deck, and $X_i = 0$ otherwise. Then $Y = X_1 + \cdots + X_{39}$ is the number of cards we have to draw before a heart appears. We are required to compute $\mathbf{E}Y = \mathbf{E}X_1 + \cdots + \mathbf{E}X_{39}$. Since each card arrangement is equally likely, $\mathbf{E}X_1 = \mathbf{E}X_2 = \cdots = \mathbf{E}X_{39}$. So, we have

$$\mathbf{E}Y = 39\mathbf{E}X_1.$$

By definition of X_1, $\mathbf{E}X_1$ is the probability that a single non-heart card appears above every heart in the deck. If we think of the thirteen heart cards as fixed in the deck, there are fourteen positions that a non-heart card can take relative to the hearts, and only one such position is above all of those hearts. So, $\mathbf{E}X_1 = 1/14$, since all arrangements of cards are equally likely. In conclusion,

$$\mathbf{E}Y = \frac{39}{14}.$$