407 Midterm 1v2 Solutions[1]

1. Question 1

TRUE/FALSE

(a) The negation of the statement "For every integer j, we have $j^2 + 1 > 0$" is: "There exists and integer j such that $j^2 + 1 \leq 0$."

TRUE, by the rules of negation, "For every" is negated to "There exists" and the inequality $>$ is negated to $\leq$.

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. For any $A \subseteq \Omega$, define $\mathbf{P}(A)$ to be the number of elements in A. Then $\mathbf{P}$ is a probability law on Ω.

FALSE. $\mathbf{P}(\Omega) = 7$, but Axiom (iii) for probability laws says $\mathbf{P}(\Omega) = 1$. So, $\mathbf{P}$ is not a probability law.

(c) Let A, B be subsets of a sample space Ω. Then

$$A = (A \cap B) \cup (A \cap B^c).$$

TRUE. Since $B \cup B^c = \Omega$, $A = A \cap \Omega = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$. (d) Let $A_1, \ldots, A_n$ be disjoint events in a sample space Ω. That is, $A_i \cap A_j = \emptyset$ whenever $i, j \in \{1, \ldots, n\}$ satisfy $i \neq j$. Let $\mathbf{P}$ be a probability law on Ω. Assume $\mathbf{P}(A_i) > 0$ for all $1 \leq i \leq n$. Let $B \subseteq \Omega$. Then

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{P}(B|A_i).$$

FALSE. Let $\Omega = \{1, 2, 3\}$, let $\mathbf{P}$ be uniform on Ω, let $A_1 = \{1\}$ and let $A_2 = \{2\}$. By Axiom (iii), $\mathbf{P}(\Omega) = 1$. $\sum_{i=1}^{2} \mathbf{P}(\Omega \cap A_i) = \sum_{i=1}^{2} \mathbf{P}(\{i\}) = 2/3$. So, $\mathbf{P}(\Omega) \neq \sum_{i=1}^{2} \mathbf{P}(B \cap A_i)$. The issue is, we also need to assume that $\cup_{i=1}^{n} A_i = \Omega$.

2. Question 2

Prove the following equality for subsets of the real line:

$$\bigcup_{n=1}^{\infty} \left(0,\ 1 - \frac{1}{n}\right] = (0, 1).$$

Solution. We first show that $\bigcup_{n=1}^{\infty} \left(0,\ 1 - \frac{1}{n}\right] \subseteq (0, 1)$. Let $x \in \bigcup_{n=1}^{\infty} \left(0,\ 1 - \frac{1}{n}\right]$. By definition of countable union, there exists $j \geq 1$ such that $x \in (0, 1 - 1/j]$. That is, $0 < x < 1 - 1/j < 1$, since $-1/j < 0$. That is, $0 < x < 1$, so that $x \in (0, 1)$. In conclusion, $\bigcup_{n=1}^{\infty} \left(0,\ 1 - \frac{1}{n}\right] \subseteq (0, 1)$.

We now show the reverse inclusion $\bigcup_{n=1}^{\infty} \left(0,\ 1 - \frac{1}{n}\right] \supseteq (0, 1)$. Let $x \in (0, 1)$. Since $x \in (0, 1)$, $x < 1$, that is, $1 - x > 0$. By the Archimedean property of the real numbers, there exists a positive integer $j \geq 1$ such that $1 - x > 1/j > 0$, i.e. $x < 1 - 1/j$. That is, $x \in (0, 1 - 1/j]$ (using also $x > 0$). By definition of countable union, we therefore have $x \in \bigcup_{n=1}^{\infty} \left(0,\ 1 - \frac{1}{n}\right]$. In conclusion, $\bigcup_{n=1}^{\infty} \left(0,\ 1 - \frac{1}{n}\right] \supseteq (0, 1)$.

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Thus we have shown that $\bigcup_{n=1}^{\infty}\Big(0,\ 1-\frac{1}{n}\Big] \supseteq (0,1)$ and $\bigcup_{n=1}^{\infty}\Big(0,\ 1-\frac{1}{n}\Big] \subseteq (0,1)$, which implies that $\bigcup_{n=1}^{\infty}\Big(0,\ 1-\frac{1}{n}\Big] = (0,1)$.

3. Question 3

A single fair 100-sided die has each of its faces labeled with exactly one integer between and including 1 and 100. Each face is equally like to be rolled.

Suppose you have three fair 100-sided dice. After rolling these three dice, what is the probability that the sum of the rolls of the three dice is 52?

Solution. For any $1 \leq i \leq 100$, let A_i be the event that that first die roll is i. Let B be the event that the sum of the rolls is 52. Then $\mathbf{P}(B) = \sum_{i=1}^{100} \mathbf{P}(B|A_i)\mathbf{P}(A_i)$, by the Total Probability Theorem. (Here we used $\cup_{i=1}^{100} A_i = \Omega$, and $A_i \cap A_j = \emptyset$ for every $i \neq j$ with $1 \leq i, j \leq 100$.) Now, $\mathbf{P}(B|A_i) = 0$ if $i > 50$, since if the first roll exceeds 50, the sum of the rolls must exceed 52, so that $B|A_i$ is empty. So, $\mathbf{P}(B) = \sum_{i=1}^{50} \mathbf{P}(B|A_i)\mathbf{P}(A_i)$. Also, $\mathbf{P}(A_i) = 1/100$ for every $1 \leq i \leq 100$ since the first die is fair, so $\mathbf{P}(B) = \frac{1}{100}\sum_{i=1}^{50} \mathbf{P}(B|A_i)$. Given that A_i occurs, the sum of the remaining two dice is $52 - i = s$. Arguing as in class (or just counting the possibilities), the probability that two of the dice sum to $s = 52 - i$ is $10^{-4}(s-1) = 10^{-4}(51-i)$. Therefore,

$$\mathbf{P}(B) = 10^{-6}\sum_{i=1}^{50}(51-i) = 10^{-6}(50 \cdot 51 - 50 \cdot 51/2) = 10^{-6}(25 \cdot 51) = \frac{51}{40000}.$$

4. Question 4

Let A, B, C, D be events in a sample space Ω. Prove:

$$\mathbf{P}(A \cap B \cap C \cap D) = \mathbf{P}(A) \cdot \mathbf{P}(B|A) \cdot \mathbf{P}(C|\, A \cap B) \cdot \mathbf{P}(D|\, A \cap B \cap C).$$

Solution. We assume that $\mathbf{P}(A) > 0$, $\mathbf{P}(A \cap B) > 0$ and $\mathbf{P}(A \cap B \cap C) > 0$, otherwise the conditional probabilities are undefined. The proof of this identity follows from the Multiplication Rule in the notes (i.e. it is proven there).

5. Question 5

Let A, B, C be subsets of a sample space Ω. Suppose we know that

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C).$$

Are the sets A, B, C independent?

If the sets A, B, C are independent, prove it.

If it is possible that the sets A, B, C are not be independent, provide a counterexample and explain your reasoning. (Make sure to specify $\Omega, \mathbf{P}, A, B$ and C.) *Solution.* It is possible that A, B, C might not be independent. Let $\Omega = \{1, 2, 3\}$, let $\mathbf{P}$ be uniform on Ω, let $A = \{1\}$ and let $B = \{2\}$ and let $C = \emptyset$. Then $\mathbf{P}(C) = 0$ and $A \cap B \cap C = \emptyset$ so $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(\emptyset) = 0 = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$. However, these sets are not independent. In order to be independent, we must have $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$. But since $\mathbf{P}$ is uniform, we have $\mathbf{P}(A \cap B) = \mathbf{P}(\emptyset) = 0 \neq (1/3)^2 = \mathbf{P}(A)\mathbf{P}(B)$.