407 Midterm 2 Version 2 Solutions[1]

1. Question 1

TRUE/FALSE

(a) Let X be a continuous random variable with probability density function f_X. Then $f_X(x) \leq 1$ for all $x \in \mathbf{R}$.

FALSE. A density function can have value larger than 1. For example, if $f_X(x) = 2$ for any $x \in [0, 1/2]$ and $f_X(x) = 0$ otherwise, then f_X is a PDF.

(b) When X is a continuous random variable, there is a continuous function $f_X\colon \mathbf{R} \to [0, \infty)$ such that, for any $-\infty \leq a \leq b \leq \infty$,

$$\mathbf{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

FALSE. f_X need not be continuous. When X is uniformly distributed in $[0, 1]$, the density f_X is discontinuous at $x = 0$ and at $x = 1$.

(c) Let X be a continuous random variable with PDF $f_X\colon \mathbf{R} \to [0, \infty)$. Then, for any $t \in \mathbf{R}$,

$$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t).$$

FALSE. When X is uniformly distributed in $[0, 1]$, the density f_X is discontinuous at $x = 0$ and at $x = 1$, and $\mathbf{P}(X \leq t)$ is not differentiable at $t = 0$, so that $\frac{d}{dt}\mathbf{P}(X \leq t)$ does not exist when $t = 0$.

(d) Let X and Y be discrete random variables. Then

$$\mathbf{E}(XY) = (\mathbf{E}X)(\mathbf{E}Y).$$

FALSE. Let X be a Bernoulli random variable with parameter $0 < p < 1$, and let $Y = X$, then $\mathbf{E}XY = \mathbf{E}X^2 = \mathbf{E}X = p$ while $\mathbf{E}X\mathbf{E}Y = (\mathbf{E}X)^2 = p^2$ and $p \neq p^2$.

(e) Let $A_1, \ldots, A_n$ be disjoint events in a sample space Ω. That is, $A_i \cap A_j = \emptyset$ whenever $i, j \in \{1, \ldots, n\}$ satisfy $i \neq j$. Let $\mathbf{P}$ be a probability law on Ω. Assume $\mathbf{P}(A_i) > 0$ for all $1 \leq i \leq n$. Let $X\colon \Omega \to \mathbf{R}$ be a discrete random variable. Then

$$\mathbf{E}X = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{E}(X|A_i).$$

FALSE. Let $\Omega = \{1, 2, 3\}$, let $\mathbf{P}$ be uniform on Ω, let $A_1 = \{1\}$ and let $A_2 = \{2\}$. Let X so that $X(\omega) = \omega$ for all $\omega \in \Omega$. Then $E(X|A_1) = 1$ and $\mathbf{E}(X|A_2) = 2$, whereas $\mathbf{E}X = (3 + 2 + 1)/3 = 2$, but

$$\sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{E}(X|A_i) = (1/3)(1 + 2) = 1 \neq 2 = \mathbf{E}X.$$

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2. Question 2

Let X be a discrete random variable such that

$$\mathbf{P}(X=1)=\mathbf{P}(X=2)=\mathbf{P}(X=3)=1/6, \qquad \text{and}$$
$$\mathbf{P}(X=-1)=\mathbf{P}(X=-2)=\mathbf{P}(X=-3)=1/6.$$

Compute the following quantities: $\mathbf{E}X$, $\mathbf{E}(X^2)$, $\mathrm{var}(X)$.

Solution. By definition of X and $\mathbf{E}X$,

$$\mathbf{E}X=\sum_{x\in\mathbf{R}}xp_X(x)=(1/6)(3+2+1-1-2-3)=0.$$

$$\mathbf{E}X^2=\sum_{x\in\mathbf{R}}x^2p_X(x)=(1/6)(3^2+2^2+1^2+(-1)^2+(-2)^2+(-3)^2)=28/6=14/3.$$

Lastly,

$$\mathrm{var}(X)=\mathbf{E}X^2-(\mathbf{E}X)^2=14/3-0=14/3.$$

3. Question 3

Let X be an exponential random variable with parameter $\lambda=1$, so that X has PDF

$$f_X(x)=e^{-x}, \qquad \forall\, x\geq 0,$$

and $f_X(x)=0$ for all $x<0$.

Compute the following quantities: $\mathbf{E}X$, $\mathbf{P}(X>1)$.

Solution. By definition of $\mathbf{E}X$,

$$\begin{aligned}\mathbf{E}X&=\int_{\mathbf{R}}xf_X(x)dx=\int_0^\infty xe^{-x}dx=\lim_{N\to\infty}\int_0^N x[-(d/dx)e^{-x}]dx\\&=\lim_{N\to\infty}[-xe^{-x}]_{x=0}^{x=N}+\int_0^N e^{-x}dx=\lim_{N\to\infty}(-Ne^{-N})+[1-e^{-N}]=1.\end{aligned}$$

$$\mathbf{P}(X>1)=\int_1^\infty e^{-x}dx=\lim_{N\to\infty}\int_1^N e^{-x}dx=\lim_{N\to\infty}e^{-1}-e^{-N}=e^{-1}.$$

4. Question 4

Let X and Y be discrete random variables such that $|X|\leq 10$ and $|Y|\leq 10$. Recall that $\mathrm{cov}(X,Y)=\mathbf{E}[(X-\mathbf{E}X)(Y-\mathbf{E}Y)]$.

Prove or disprove the statement below. (In the case that you disprove the statement, it suffices to find a counterexample and explain your reasoning.)

Statement: If $\mathrm{cov}(X,Y)=0$, then X and Y are independent.

Solution. This statement is false. Let X be uniformly distributed in $\{-1,0,1\}$ and let $Y=|X|$. Then $\mathbf{E}X=(1/3)(1+0+-1)=0$, and

$$\mathbf{E}XY=\mathbf{E}X\,|X|=(1/3)(1\cdot 1)+(1/3)(0)+(1/3)(-1\cdot 1)=0.$$

$$\mathbf{E}X\mathbf{E}Y=0.$$

So, $\mathrm{cov}(X,Y)=\mathbf{E}XY-\mathbf{E}X\mathbf{E}Y=0$. However, X and Y are not independent, since

$$\mathbf{P}(X=1,Y=1)=\mathbf{P}(X=1)=1/3\neq(1/3)(2/3)=\mathbf{P}(X=1)\mathbf{P}(Y=1).$$

5. Question 5

Let X be a standard Gaussian random variable, so that X has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \forall\, x \in \mathbf{R}.$$

Let Z be the random variable defined by

$$Z = X^4.$$

What is the PDF of Z? (As usual, justify your answer.)

Solution. Let $t > 0$. Then

$$\mathbf{P}(Z \leq t) = \mathbf{P}(X^4 \leq t) = \mathbf{P}(X \leq t^{1/4}) = \int_{-t^{1/4}}^{t^{1/4}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 2\int_0^{t^{1/4}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

So, by the Chain Rule and the Fundamental Theorem of Calculus,

$$f_Z(t) = \frac{d}{dt}\mathbf{P}(Z \leq t) = 2(1/4)t^{-3/4} \frac{1}{\sqrt{2\pi}} e^{-(t^{1/4})^2/2} = (1/2)t^{-3/4} \frac{1}{\sqrt{2\pi}} e^{-t^{1/2}/2}.$$