

Please provide complete and well-written solutions to the following exercises.

Due March 30, 2PM PST, to be uploaded in blackboard as a single PDF document (in the Assignments tab).

Homework 8

Exercise 1. Let X be a continuous random variable with distribution function $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \forall x \in \mathbf{R}$. Show that $\text{var}(X) = 1$.

Exercise 2. Let X be a random variable such that $f_X(x) = x$ when $0 \leq x \leq \sqrt{2}$ and $f_X(x) = 0$ otherwise. Compute $\mathbf{E}X^2$ and $\mathbf{E}X^3$.

Exercise 3 (Numerical Integration). In computer graphics in video games, etc., various integrations are performed in order to simulate lighting effects. Here is a way to use random sampling to integrate a function in order to quickly and accurately render lighting effects. Let $\Omega = [0, 1]$, and let $\mathbf{P}$ be the uniform probability law on Ω , so that if $0 \leq a < b \leq 1$, we have $\mathbf{P}([a, b]) = b - a$. Let $X_1, \dots, X_n$ be independent random variables such that $\mathbf{P}(X_i \in [a, b]) = b - a$ for all $0 \leq a < b \leq 1$, for all $i \in \{1, \dots, n\}$. Let $f: [0, 1] \rightarrow \mathbf{R}$ be a continuous function we would like to integrate. Instead of integrating f directly, we instead compute the quantity

$$\frac{1}{n} \sum_{i=1}^n f(X_i).$$

Show that

$$\lim_{n \rightarrow \infty} \mathbf{E} \left(\frac{1}{n} \sum_{i=1}^n f(X_i) \right) = \int_0^1 f(t) dt.$$

$$\lim_{n \rightarrow \infty} \text{var} \left(\frac{1}{n} \sum_{i=1}^n f(X_i) \right) = 0.$$

That is, as n becomes large, $\frac{1}{n} \sum_{i=1}^n f(X_i)$ is a good estimate for $\int_0^1 f(t) dt$.

Exercise 4. Let X be a random variable such that $X = 1$ with probability 1. Show that X is not a continuous random variable. That is, there does not exist a probability density function f such that $\mathbf{P}(X \leq a) = \int_{-\infty}^a f(x) dx$ for all $x \in \mathbf{R}$. (Hint: if X were continuous, then the function $g(t) = \int_{-\infty}^t f(x) dx$ would be continuous, by the Fundamental Theorem of Calculus.)

Exercise 5. Verify that a Gaussian random variable X with mean μ and variance σ^2 actually has mean μ and variance σ^2 .

Let $a, b \in \mathbf{R}$ with $a \neq 0$. Show that $aX + b$ is a normal random variable with mean $a\mu + b$ and variance $a^2\sigma^2$.

In particular, conclude that $(X - \mu)/\sigma$ is a standard normal.

Exercise 6. Using the De Moivre-Laplace Theorem, estimate the probability that 1,000,000 coin flips of fair coins will result in more than 501,000 heads. (Some of the following integrals may be relevant: $\int_{-\infty}^0 e^{-t^2/2} dt / \sqrt{2\pi} = 1/2$, $\int_{-\infty}^1 e^{-t^2/2} dt / \sqrt{2\pi} \approx .8413$, $\int_{-\infty}^2 e^{-t^2/2} dt / \sqrt{2\pi} \approx .9772$, $\int_{-\infty}^3 e^{-t^2/2} dt / \sqrt{2\pi} \approx .9987$.)

Casinos do these kinds of calculations to make sure they make money and that they do not go bankrupt. Financial institutions and insurance companies do similar calculations for similar reasons.

Exercise 7. Let X be a uniformly distributed random variable on $[-1, 1]$. Let $Y := X^2$. Find f_Y .

Exercise 8. Let X be a uniformly distributed random variable on $[0, 1]$. Let $Y := 4X(1 - X)$. Find f_Y .

Exercise 9. Let X be a uniformly distributed random variable on $[0, 1]$. Find the PDF of $-\log(X)$.

Exercise 10. Let X be a standard normal random variable. Find the PDF of e^X .