Please provide complete and well-written solutions to the following exercises.

Due April 6, 2PM PST, to be uploaded in blackboard as a single PDF document (in the Assignments tab).

Homework 9

Exercise 1. Let X, Y be random variables with joint PDF $f_{X,Y}$. Let $a, b \in \mathbf{R}$. Using the definition of expected value, show that $\mathbf{E}(aX + bY) = a\mathbf{E}X + b\mathbf{E}Y$.

Exercise 2. Let X_1, Y_1 be random variables with joint PDF f_{X_1,Y_1}. Let X_2, Y_2 be random variables with joint PDF f_{X_2,Y_2}. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ and let $S\colon \mathbf{R}^2 \to \mathbf{R}^2$ so that $ST(x,y) = (x,y)$ and $TS(x,y) = (x,y)$ for every $(x,y) \in \mathbf{R}^2$. Let $J(x,y)$ denote the determinant of the Jacobian of S at (x,y). Assume that $(X_2, Y_2) = T(X_1, Y_1)$. Using the change of variables formula from multivariable calculus, show that

$$f_{X_2,Y_2}(x,y) = f_{X_1,Y_1}(S(x,y)) \left|J(x,y)\right|.$$

Exercise 3. Let X and Y be nonnegative random variables. Recall that we can define

$$\mathbf{E}X := \int_0^\infty \mathbf{P}(X > t)dt.$$

Assume that $X \leq Y$. Conclude that $\mathbf{E}X \leq \mathbf{E}Y$.

More generally, if X satisfies $\mathbf{E}\left|X\right| < \infty$, we define $\mathbf{E}X := \mathbf{E}\max(X,0) - \mathbf{E}\max(-X,0)$. If X, Y are any random variables with $X \leq Y$, $\mathbf{E}\left|X\right| < \infty$ and $\mathbf{E}\left|Y\right| < \infty$, show that $\mathbf{E}X \leq \mathbf{E}Y$.

Exercise 4. Let X, Y, Z be independent standard Gaussian random variables. Find the PDF of $\max(X, Y, Z)$.

Exercise 5. Let X be a random variable uniformly distributed in $[0,1]$ and let Y be a random variable uniformly distributed in $[0,2]$. Suppose X and Y are independent. Find the PDF of X/Y^2.

Exercise 6. Let X, Y be independent random variables with joint PDF $f_{X,Y}$. Show that

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y).$$

Exercise 7. Let X and Y be uniformly distributed random variables on $[0,1]$. Assume that X and Y are independent. Compute the following probabilities:

- $\mathbf{P}(X > 3/4)$
- $\mathbf{P}(Y < X)$
- $\mathbf{P}(X + Y < 1/2)$
- $\mathbf{P}(\max(X,Y) > 1/2)$
- $\mathbf{P}(XY < 1/3)$.

Exercise 8. Let X, Y be random variables with $\mathbf{E}X^2 < \infty$ and $\mathbf{E}Y^2 < \infty$. Prove the **Cauchy-Schwarz inequality**:

$$\mathbf{E}(XY) \leq (\mathbf{E}X^2)^{1/2}(\mathbf{E}Y^2)^{1/2}.$$

Then, deduce the following when X, Y both have finite variance:

$$|\mathrm{cov}(X,Y)| \leq (\mathrm{var}(X))^{1/2}(\mathrm{var}(Y))^{1/2}.$$

(Hint: in the case that $\mathbf{E}Y^2 > 0$, expand out the product $\mathbf{E}(X - Y\mathbf{E}(XY)/\mathbf{E}Y^2)^2$.)

Exercise 9. Suppose you go to the bus stop, and the time T between successive arrivals of the bus is anything between 0 and 30 minutes, with all arrival times being equally likely.

Suppose you get to the bus stop, and the bus just leaves as you arrive. How long should you expect to wait for the next bus? What is the probability that you will have to wait at least 15 minutes for the next bus to arrive?

On a different day, suppose you go to the bus stop and someone says the last bus came 10 minutes ago. How long should you expect to wait for the next bus? What is the probability that you will have to wait at least 10 minutes for the next bus to arrive?

Exercise 10. Let $A_1, A_2, \ldots$ be disjoint events such that $\mathbf{P}(A_i) = 2^{-i}$ for each $i \geq 1$. Assume that $\cup_{i=1}^{\infty} A_i = \Omega$. Let X be a random variable such that $\mathbf{E}(X|A_i) = (-1)^{i+1}$ for each $i \geq 1$. Compute $\mathbf{E}X$.

Exercise 11. Let X, Y be random variables. For any $y \in \mathbf{R}$, assume that $\mathbf{E}(X|Y = y) = e^{-|y|}$. Also, assume that Y has an exponential distribution with parameter $\lambda = 2$. Compute $\mathbf{E}X$.