Math 407, Fall 2023, USC — Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ____________

Signature: ______________________. Discussion Section: ______________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 1

This exam contains 7 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Let A be a subset of some universal set Ω. Prove the following:

$$(A^c)^c = A.$$

(Recall that $A^c = \{x \in \Omega \colon x \notin A\}$.)

2. (10 points) Suppose you roll four distinct, fair, three-sided dice.

   What is the probability that the sum of the dice rolls is 5?

   (Each die has three sides, and these three sides are each labelled with distinct integers among $\{1, 2, 3\}$.)

3. (10 points) Prove the following assertion by induction on n:

For any positive integer $n \geq 1$, we have

$$\sum_{k=1}^{n} (1/3)^k = \frac{1 - \frac{1}{3^n}}{2}$$

(You MUST use induction to prove this assertion.)

4. (10 points) You are a contestant on a game show. There are five doors labelled $1, 2, 3, 4$ and 5. You and the host are aware that one door contains a prize, and the four other doors have no prize. The host knows where the prize is, but you do not. Each door is equally likely to contain a prize, i.e. each door has a $1/5$ chance of containing the prize. In the first step of the game, you can select one of the five doors. Suppose the selected door is $i \in \{1, 2, 3, 4, 5\}$. Given your selection, the host now reveals three of the four remaining doors, demonstrating that those doors contain no prize. The game now concludes with a choice. You can either keep your current door i, or you can switch to the other unopened door. You receive whatever is behind your selected door.

   The question is: should you switch your door choice or not?

   (Your goal is to win the prize.)

5. (10 points) An urn contains four red cubes and two blue cubes. A cube is removed from the urn uniformly at random. If the cube is red, it is kept out of the urn and a second cube is removed from the urn. If the cube is blue, then this cube is put back into the urn and an additional two blue cubes are put into the urn, and then a second cube is removed from the urn.

   - What is the probability that the second cube removed from the urn is red?
   - If it is given information that the second cube removed from the urn is red, then what is the probability that the first cube removed from the urn is blue?

(Scratch paper)