407 Midterm 1 Solutions[1]

1. Question 1

Let A be a subset of some universal set Ω. Prove the following:

$$(A^c)^c = A.$$

Solution. Let $x \in A$. Then $x \notin \Omega \setminus A$, i.e. $x \notin A^c$, by definition of complement. Using the definition of complement again, $x \in (A^c)^c$. We have therefore shown that $A \subseteq (A^c)^c$. It therefore remains to show: if $x \in (A^c)^c$, then $x \in A$. Suppose $x \in (A^c)^c$. By definition of complement, this means $x \in A^c$. By definition of complement again, this means $x \in A$. Having shown $A \subseteq (A^c)^c$ and $A \supseteq (A^c)^c$, we conclude that $A = (A^c)^c$.

2. Question 2

Suppose you roll four distinct, fair, three-sided dice.

What is the probability that the sum of the dice rolls is 5?

(Each die has three sides, and these three sides are each labelled with distinct integer among $\{1, 2, 3\}$.)

Solution. The only way the dice can sum to 5 is to roll the numbers $\{2, 1, 1, 1\}$. (Since each roll is at least 1, rolling a 3 or greater is impossible, since doing so would make the sum of the rolls at least $3 + 1 + 1 + 1 = 6$. Also, rolling all 1's results in a total roll sum of four. And rolling at least two 2's is impossible, since doing so would make the sum of the rolls at least $2 + 2 + 1 + 1 = 6$. So, the only possible roll outcomes are $\{2, 1, 1, 1\}$.) There are only four possible outcomes where a reordering of this list of numbers occurs:

$$\{(2, 1, 1, 1), \quad (1, 2, 1, 1), \quad (1, 1, 2, 1), \quad (1, 1, 1, 2)\}.$$

There are 3^4 possible combinations of dice rolls, all equally likely by assumption. So, the probability that the sum of the dice rolls is 5 is:

$$\frac{4}{3^4} = \frac{4}{81} \approx .049.$$

3. Question 3

Prove the following assertion by induction on n:

For any positive integer $n \geq 1$, we have

$$\sum_{k=1}^{n} (1/3)^k = \frac{1 - \frac{1}{3^n}}{2}$$

(You MUST use induction to prove this assertion.)

Solution. In the base case $n = 1$, the left side is equal to $(1/3)$, and the right side is equal to $(2/3)/2 = 1/3$. so, the equality holds in the case $n = 1$, i.e. the base case holds. We now

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prove the inductive step. Assume the above equality holds for some $n \geq 1$, and we need to show it holds in the case $n+1$. We have by the inductive hypothesis that

$$\sum_{k=1}^{n+1}(1/3)^k = (1/3)^{n+1} + \sum_{k=1}^{n}(1/3)^k = (1/3)^{n+1} + \frac{1-\frac{1}{3^n}}{2} = \frac{1-\frac{1}{3^n}+(2/3)(1/3)^n}{2}$$
$$= \frac{1-(1/3)\frac{1}{3^n}}{2} = \frac{1-\frac{1}{3^{n+1}}}{2}.$$

That is, we have verified the above equality holds in the case $n+1$. Having completed the inductive step, the assertion must hold for all $n \geq 1$.

4. Question 4

You are a contestant on a game show. There are five doors labelled $1, 23, 4$ and 5. You and the host are aware that one door contains a prize, and the four other doors have no prize. The host knows where the prize is, but you do not. Each door is equally likely to contain a prize, i.e. each door has a $1/5$ chance of containing the prize. In the first step of the game, you can select one of the five doors. Suppose the selected door is $i \in \{1, 2, 3, 4, 5\}$. Given your selection, the host now reveals three of the four remaining doors, demonstrating that those doors contain no prize. The game now concludes with a choice. You can either keep your current door i, or you can switch to the other unopened door. You receive whatever is behind your selected door.

The question is: should you switch your door choice or not?

Solution. At the beginning of the game, the probability you select the correct door is $1/5$. In this case, you should not switch your door choice. The probability you do not select the correct door at the start of the game is $4/5$. Since the only unrevealed door in this case has the prize, you should switch your door choice. That is, with probability $4/5$, you should switch your door choice at the end of the game.

5. Question 5

An urn contains four red cubes and two blue cubes. A cube is removed from the urn uniformly at random. If the cube is red, it is kept out of the urn and a second cube is removed from the urn. If the cube is blue, then this cube is put back into the urn and an additional two blue cubes are put into the urn, and then a second cube is removed from the urn.

- What is the probability that the second cube removed from the urn is red?
- If it is given information that the second cube removed from the urn is red, then what is the probability that the first cube removed from the urn is blue?

Solution. Let A be the event that the first cube removed is red, and let B be the event that the first cube removed is blue. Let C be the event that the second cube removed from the urn is red. Then $A \cap B = \emptyset$ and $A \cup B = \Omega$, so the Total Probability Theorem says

$$\mathbf{P}(C) = \mathbf{P}(C|A)\mathbf{P}(A) + \mathbf{P}(C|B)\mathbf{P}(B) = (3/5)(4/6) + (4/8)(2/6) = 12/30 + 1/6 = 17/30.$$

Now, using that $\mathbf{P}(C) = 17/30$, we have

$$\mathbf{P}(B|C) = \mathbf{P}(C|B)[\mathbf{P}(B)/\mathbf{P}(C)] = (4/8)(2/6)(30/17) = 5/17.$$