Math 407, Fall 2023, USC Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________. Discussion Section: ______________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 2

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck![a]

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (2 points) There is some random variable X such that $\mathrm{var}(X) = -1$.

   TRUE FALSE (circle one)

   (b) (2 points) Let A, B, C be events in a sample space. Assume that

   $$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B), \quad \mathbf{P}(A \cap C) = \mathbf{P}(A)\mathbf{P}(C), \quad \mathbf{P}(B \cap C) = \mathbf{P}(B)\mathbf{P}(C).$$

   Then the events A, B, C are independent.
   (For this question, you can freely use a result from the homework.)

   TRUE FALSE (circle one)

   (c) (2 points) Let X, Y, Z be discrete random variables such that

   $$\mathbf{P}(X = x, Y = y, Z = z) = \mathbf{P}(X = x)\mathbf{P}(Y = y)\mathbf{P}(Z = z), \qquad \forall\, x, y, z \in \mathbf{R}.$$

   Then the random variable X is independent of the random variable Z.

   TRUE FALSE (circle one)

(d) (2 points) Let X be a continuous random variable. Let f_X be the probability density function of X. Then f_X is a continuous function.

TRUE FALSE (circle one)

(e) (2 points) Let X be a continuous random variable. Then, for any $t \in \mathbf{R}$, the derivative $\frac{d}{dt}\mathbf{P}(X \leq t)$ exists (i.e. the function $\mathbf{P}(X \leq t)$ is differentiable at each $t \in \mathbf{R}$). Also, the probability density function f_X of X satisfies

$$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t), \qquad \forall\, t \in \mathbf{R}.$$

TRUE FALSE (circle one)

2. (10 points) Let X be a random variable such that

$$\mathbf{P}(X = -2) = \mathbf{P}(X = 1) = \mathbf{P}(X = 3) = \mathbf{P}(X = 4) = 1/4.$$

Compute the following quantities:

- $\mathbf{E}X$
- $\mathbf{E}(X^2)$
- $\mathrm{var}(X)$

Simplify your answers to the best of your ability. (As usual, show your work.)

3. (10 points) Let X, Y be random variables with joint PMF $p_{X,Y}$ such that

$$p_{X,Y}(x, y) = \mathbf{P}(X = x, Y = y) = 1/9, \quad \text{for all integers} \quad -1 \leq x, y \leq 1$$

Compute the following quantities:

- $\mathbf{P}(X = 1)$.
- $\mathbf{E}(Y \mid X = 1)$
- $\mathbf{E}X^2$

Simplify your answers to the best of your ability. (As usual, show your work.)

4. (10 points) Let X be an exponential random variable with parameter 1 so that X has PDF

$$f_X(x) = \begin{cases} e^{-x} & , \text{if } x \geq 0. \\ 0 & , \text{if } x < 0. \end{cases}$$

You can freely use that $\mathbf{E}X = 1$ and $\mathrm{Var}(X) = 1$.

Compute the following quantities:

- $\mathbf{P}(X = 3)$.
- $\mathbf{P}(X > 1)$.
- $\mathrm{Var}(10X + 175)$.

Simplify your answers to the best of your ability. (As usual, show your work.)

5. (10 points) Suppose there are five separate bins. You first place a sphere randomly in one of the bins, where each bin has an equal probability of getting the sphere. Once again, you randomly place another sphere uniformly at random in one of the bins. This process occurs twenty times, so that twenty spheres have been placed in bins. (All of the sphere placements up to this point are independent of each other).

   Suppose you now flip a fair coin. (A fair coin has probability $1/2$ of landing heads, and probability $1/2$ of landing tails). (The coin flip result is independent of all of the sphere placements.) If the coin lands heads, you then place another ten spheres randomly into the bins (with each sphere being equally likely to appear in any of the five bins).

   What is the expected number of empty bins?

   Simplify your answer to the best of your ability. (As usual, show your work.)

(Scratch paper)