407 Midterm 2 Solutions[1]

1. Question 1

(a)There is some random variable X such that $\mathrm{var}(X) = -1$.
False. Since $(X - \mathbf{E}X)^2 \geq 0$, we conclude that $\mathrm{var}(X) = \mathbf{E}(X - \mathbf{E}X)^2 \geq \mathbf{E}0 = 0$.
(b)Let A, B, C be events in a sample space. Assume that

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B), \quad \mathbf{P}(A \cap C) = \mathbf{P}(A)\mathbf{P}(C), \quad \mathbf{P}(B \cap C) = \mathbf{P}(B)\mathbf{P}(C).$$

Then the events A, B, C are independent.
(For this question, you can freely use a result from the homework.)

FALSE. We showed on the homework that there are sets A, B, C that are pairwise independent (i.e. they satisfy the above assumption), but A, B, C are not independent, i.e. $\mathbf{P}(A \cap B \cap C) \neq \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$

(c) Let X, Y, Z be discrete random variables such that

$$\mathbf{P}(X = x, Y = y, Z = z) = \mathbf{P}(X = x)\mathbf{P}(Y = y)\mathbf{P}(Z = z), \qquad \forall\, x, y, z \in \mathbf{R}.$$

Then the random variable X is independent of the random variable Z.

TRUE. We have assumed that X, Y, Z are all independent. It follows from a proposition in the notes that X and Z are independent.

(d) Let X be a continuous random variable. Let f_X be the probability density function of X. Then f_X is a continuous function.

FALSE. For example, when X is uniformly distributed on $[0, 1]$, the PDF f_X is not continuous at the point $x = 0$.

(e) Let X be a continuous random variable. Then, for any $t \in \mathbf{R}$, the derivative $\frac{d}{dt}\mathbf{P}(X \leq t)$ exists (i.e. the function $\mathbf{P}(X \leq t)$ is differentiable at each $t \in \mathbf{R}$). Also, the probability density function f_X of X satisfies

$$\frac{d}{dt}\mathbf{P}(X \leq t) = f_X(t), \qquad \forall\, t \in \mathbf{R}.$$

FALSE. For example, when X is uniformly distributed on $[0, 1]$, the CDF of X is not differentiable at the point $x = 0$.

2. Question 2

Let X be a random variable such that

$$\mathbf{P}(X = -2) = \mathbf{P}(X = 1) = \mathbf{P}(X = 3) = \mathbf{P}(X = 4) = 1/4.$$

Compute the following quantities:

- $\mathbf{E}X$
- $\mathbf{E}(X^2)$
- $\mathrm{var}(X)$

Simplify your answers to the best of your ability. (As usual, show your work.)

Solution. By definition of $\mathbf{E}X$ we have $\mathbf{E}X = (1/4)(-2 + 1 + 3 + 4) = (1/4)(6) = 3/2$. Similarly, $\mathbf{E}X^2 = (1/4)(4+1+9+16) = 15/2$. And $\mathrm{Var}(X) = \mathbf{E}X^2 - (\mathbf{E}X)^2 = 15/2 - 9/2 = 6/2 = 3$.

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3. Question 3

Let X, Y be random variables with joint PMF $p_{X,Y}$ such that

$$p_{X,Y}(x,y) = \mathbf{P}(X = x, Y = y) = 1/9, \quad \text{for all integers} \quad -1 \leq x, y \leq 1$$

Compute the following quantities:

- $\mathbf{P}(X = 1)$.
- $\mathbf{E}(Y \mid X = 1)$
- $\mathbf{E}X^2$

Simplify your answers to the best of your ability. (As usual, show your work.)

Solution. First, note that $p_X(x) = \sum_{y=1}^{3} p_{X,Y}(x,y) = 1/3$ for every integer $-1 \leq x \leq 1$, by Proposition 4.17 in the notes. So, $\mathbf{P}(X > 2) = p_X(3) = 1/3$.

Now, by definition of conditional expectation, we have $\mathbf{P}(Y = y|X = 1) = \mathbf{P}(Y = y, X = 1)/\mathbf{P}(X = 1) = (1/9)/(1/3)$ if $y \in \{-1, 0, 1\}$, so that

$$\mathbf{E}(Y|X = 1) = \sum_{y \in \mathbf{R}} y p_{Y|X=1}(y|1) = (-1)(1/3) + 0(1/3) + 1(1/3) = 0.$$

Finally, since we found p_X, we can compute $\mathbf{E}X^2 = \sum_{x \in \mathbf{R}} x^2 p_X(x) = 1p_X(-1) + 0p_X(0) + 1p_X(1) = 2/3$.

4. Question 4

Let X be an exponential random variable with parameter 1 so that X has PDF

$$f_X(x) = \begin{cases} e^{-x} & \text{, if } x \geq 0. \\ 0 & \text{, if } x < 0. \end{cases}$$

You can freely use that $\mathbf{E}X = 1$ and $\mathrm{Var}(X) = 1$.

Compute the following quantities:

- $\mathbf{P}(X = 3)$.
- $\mathbf{P}(X > 1)$.
- $\mathrm{Var}(10X + 175)$.

Simplify your answers to the best of your ability. (As usual, show your work.)

Solution. By definition of PDF, we have

$$\mathbf{P}(X = 3) = \mathbf{P}(3 \leq X \leq 3) = \int_3^3 f_X(x)dx = 0.$$

Similarly, by definition of PDF, we have

$$\mathbf{P}(X > 1) = \int_1^\infty f_X(x)dx = \int_1^\infty e^{-x}dx = \lim_{n \to \infty} [-e^{-x}]_{x=1}^{x=n} = \lim_{n \to \infty} [e^{-n} + e^{-1}] = e^{-1}.$$

Finally, using properties of variance, we have

$$\mathrm{Var}(10X + 175) = \mathrm{Var}(10X) = 100\mathrm{Var}(X) = 100.$$

5. Question 5

Suppose there are five separate bins. You first place a sphere randomly in one of the bins, where each bin has an equal probability of getting the sphere. Once again, you randomly place another sphere uniformly at random in one of the bins. This process occurs twenty times, so that twenty spheres have been placed in bins. (All of the sphere placements up to this point are independent of each other).

Suppose you now flip a fair coin. (A fair coin has probability 1/2 of landing heads, and probability 1/2 of landing tails). (The coin flip result is independent of all of the sphere placements.) If the coin lands heads, you then place another ten spheres randomly into the bins (with each sphere being equally likely to appear in any of the five bins).

What is the expected number of empty bins?

Simplify your answer to the best of your ability. (As usual, show your work.)

Solution. Let A be the event that the coin flip is heads. Let N be the number of empty bins. We are required to compute $\mathbf{E}N$. From the Total Expectation Theorem,

$$\mathbf{E}N = \mathbf{E}(N|A)\mathbf{P}(A) + \mathbf{E}(N|A^c)\mathbf{P}(A^c).$$

By its definition, $\mathbf{P}(A) = \mathbf{P}(A^c) = 1/2$, so

$$\mathbf{E}N = (1/2)[\mathbf{E}(N|A) + \mathbf{E}(N|A^c)].$$

For each $1 \leq i \leq 5$, let X_i be 1 if bin i is empty, and $X_i = 0$ otherwise. Then $N = \sum_{i=1}^5 X_i$ and $\mathbf{E}(N|A) = \sum_{i=1}^5 \mathbf{E}(X_i|A)$, $\mathbf{E}(N|A^c) = \sum_{i=1}^5 \mathbf{E}(X_i|A^c)$. Since X_i only takes values 0 or 1, we then have

$$\mathbf{E}(N|A) = \sum_{i=1}^5 \mathbf{P}(X_i = 1|A), \qquad \mathbf{E}(N|A^c) = \sum_{i=1}^5 \mathbf{P}(X_i = 1|A^c)$$

If A occurs, there are thirty total spheres placed in the bins, with all placements being independent and uniformly random. Since all sphere placements are equally likely, $\mathbf{P}(X_i = 1|A)$ is the probability that all thirty spheres lie in the other bins (other than the i^{th} bin), i.e. this probability is $(4/5)^{30}$. Similarly, if A^c occurs, there are twenty total spheres placed in the bins, with all placements being independent and uniformly random. Since all sphere placements are equally likely, $\mathbf{P}(X_i = 1|A^c)$ is the probability that all twenty spheres lie in the other bins, i.e. this probability is $(4/5)^{20}$. Combining the above, we have

$$\mathbf{E}(N|A) = \sum_{i=1}^5 (4/5)^{30} = 5(4/5)^{30}, \qquad \mathbf{E}(N|A^c) = \sum_{i=1}^5 (4/5)^{20} = 5(4/5)^{20}.$$

$$\mathbf{E}N = (1/2)[\mathbf{E}(N|A) + \mathbf{E}(N|A^c)] = (1/2)[5(4/5)^{30} + 5(4/5)^{20}].$$