Math 408, Fall 2021, USC — Instructor: Steven Heilman

Name: ____________________ USC ID: ______________ Date: ____________

Signature: ____________________. Discussion Section: ____________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 1

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	8	
3	10	
4	10	
5	10	
Total:	48	

Do not write in the table to the right. Good luck![a]

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1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample and explain your reasoning**.

   (a) (2 points) The negation of the statement
   "There exists an integer j such that $j^3 - j < 7$" is:
   "For every integer j, we have $j^3 - j \geq 7$."

   TRUE FALSE (circle one)

   (b) (2 points) Let $\mathbf{P}$ be the uniform probability law on $[0, 1]$. Let $x_1, x_2, \ldots \in [0, 1]$ be a countable set of distinct points. Then

   $$\mathbf{P}\left(\cup_{n=1}^{\infty}\{x_n\}\right) = 0.$$

   TRUE FALSE (circle one)

   (c) (2 points) Let $X_1, \ldots, X_n$ be i.i.d random variables drawn from a family of probability density functions $\{f_\theta \colon \theta \in \mathbf{R}\}$ where $f_\theta \colon \mathbf{R} \to [0, \infty)$ for all $\theta \in \mathbf{R}$. Then there must exist some integer $k \geq 1$, $\exists$ some function $t \colon \mathbf{R}^n \to \mathbf{R}^k$ and there exists some statistic $Y = t(X_1, \ldots, X_n)$ such that Y is a sufficient statistic for θ.

   TRUE FALSE (circle one)

(d) (2 points) Let $X_1, \ldots, X_8$ be i.i.d Gaussian distributed random variables. Define $W := \sum_{i=1}^{8} X_i$ and let $Z := \sqrt{\sum_{i=1}^{8}(X_i - W/8)^2}$. Then W and Z are independent.

TRUE FALSE (circle one)

(e) (2 points) Let $Y_1, Y_2, \ldots$ be a sequence of estimators that are consistent for θ, where θ lies in a parameter space Θ. That is, $Y_1, Y_2, \ldots$ converges in probability to the constant random variable θ. Then $\mathbf{E}_\theta Y_n = \theta$ for all $n \geq 1$ and for all $\theta \in \Theta$.

TRUE FALSE (circle one)

2. (8 points) Let $Y_1, Y_2, \ldots$ be random variables such that $\sqrt{n}Y_n$ converges in distribution to a mean zero Gaussian random variable with variance 5 as $n \to \infty$. Let

$$f(t) := (t+1)^3, \qquad \forall\, t \in \mathbf{R}$$

Show that, as $n \to \infty$, the random variables

$$\sqrt{n}(f(Y_n) - f(0))$$

converge in distribution to a random variable Z, and then compute $\mathbf{E}Z^2$.

3. (10 points) Let θ be a an unknown real parameter, and suppose a random variable X has PDF

$$f(x) := \begin{cases} 1 & , \text{ if } \theta \leq x \leq \theta + 1 \\ 0 & , \text{ otherwise.} \end{cases}$$

- Find a method of moments estimator for θ.
- Find a method of moments estimator for θ^2.

4. (10 points) Let $X_1, \ldots, X_n$ be a random sample of size n from a Bernoulli distribution with parameter $0 < \theta < 1$. (That is, $\mathbf{P}(X_1 = 1) = \theta$ and $\mathbf{P}(X_1 = 0) = 1 - \theta$, with θ unknown.)

   Show that $Y := X_1 + \cdots + X_n$ is a sufficient statistic for θ.

5. (10 points) Consider a population of 30,000 people, where half of them are given a vaccine for a disease. Suppose all 30,000 people are exposed to a virus causing the disease. We observe that 90 of the unvaccinated people catch the disease, while 5 of the vaccinated people catch the disease.

   Consider the following statement:

   "If we have a population of 30,000 people exposed to the virus, with half of them vaccinated, then the number of infections of vaccinated people, divided by the number of infections of unvaccinated people, is less than 15/100."

   Is the statement true with greater than 90% certainty? Justify your answer.

   (Assume that each person's ability to catch the disease is independent of each other person's ability to catch the disease.)

   (Hint: the estimated probability of a vaccinated person getting the disease is $5/15,000$, and the estimated probability of an unvaccinated person getting the disease is $90/15,000$.)

   (Hint: use the Central Limit Theorem. If Z is a standard Gaussian, then $\mathbf{P}(|Z| \leq 2) \approx .9545$. Also, $\sqrt{5} \approx 2.23$, $\sqrt{90} \approx 9.5$.)

(Scratch paper)