Please provide complete and well-written solutions to the following exercises.

Due November 16, 12PM noon PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

# Homework 6

**Exercise 1.** Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with unknown location parameter $\theta > 0$, i.e. $X_1$ has density

$$g(x) := 1_{x \geq \theta} e^{-(x-\theta)}, \qquad \forall\, x \in \mathbf{R}.$$

Fix $\theta_0 \in \mathbf{R}$. Suppose we want to test that hypothesis $H_0$ that $\theta \leq \theta_0$ versus the alternative $H_1$ that $\theta > \theta_0$. That is, $\Theta = \mathbf{R}$, $\Theta_0 = \{\theta \in \mathbf{R} \colon \theta \leq \theta_0\}$ and $\Theta_0^c = \Theta_1 = \{\theta \in \mathbf{R} \colon \theta > \theta_0\}$.

- Explicitly describe the rejection region of the generalized likelihood ratio test for this hypothesis. (Hint: it might be easier to describe the region using $x_{(1)} = \min(x_1, \ldots, x_n)$.)
- (Optional) If $H_0$ is true, then does
$$2 \log \frac{\sup_{\theta \in \Theta} f_\theta(X_1, \ldots, X_n)}{\sup_{\theta \in \Theta_0} f_\theta(X_1, \ldots, X_n)}$$
converge in distribution to a chi-squared distribution as $n \to \infty$?

**Exercise 2.** Let $X_1, \ldots, X_n$ be a random sample from a Gaussian random variable with unknown mean $\mu \in \mathbf{R}$ and unknown variance $\sigma^2 > 0$.

Fix $\mu_0 \in \mathbf{R}$. Suppose we want to test that hypothesis $H_0$ that $\mu = \mu_0$ versus the alternative $H_1$ that $\mu \neq \mu_0$.

- Explicitly describe the rejection region of the generalized likelihood ratio test for this hypothesis.
- Give an explicit formula for the $p$-value of this hypothesis test. (Hint: If $S^2$ denotes the sample variance and $\overline{X}$ denotes the sample mean, you should then be able to use the statistic $\frac{(\overline{X}-\mu_0)^2}{S^2}$. Since we have an explicit formula for Snedecor's distribution, you should then be able to write an explicit integral formula for the $p$-value of this test.)

**Exercise 3.** Write down the generalized likelihood ratio estimate for the following alpha particle data, as we did in class for a slightly different data set. The corresponding test treats individual counts of alpha particles as independent Poisson random variables, versus the alternative that the probability of a count appearing in each box of data is a sequence of nonnegative numbers that sum to one. (In doing so, you should need to compute a maximum likelihood estimate using a computer.)

| $m$ | 0, 1 or 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | $\geq 17$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # of Intervals | 16 | 26 | 58 | 102 | 125 | 146 | 163 | 164 | 120 | 100 | 72 | 54 | 20 | 12 | 10 | 4 |

Plot the MLE for the Poisson statistic (i.e. plot the denominator of the generalized likelihood ratio test statistic $\frac{\sup_{\theta\in\Theta} f_\theta(X)}{\sup_{\theta\in\Theta_0} f_\theta(X)}$) as a function of $\lambda$.

Finally, compute the value $s$ of Pearson's chi-squared statistic $S$, and compute the probability that $S \geq s$ (assuming $H_0$ holds). Does the probability $\mathbf{P}(S \geq s)$ give you confidence that the null hypothesis is true?