Quiz 2 occurs September 14, in the discussion section. The quiz will be based upon the problems below.

Quiz 2 Problems

Exercise 1. Let $X\colon \Omega \to \mathbf{R}$ be a random variable on a sample space Ω equipped with a probability law $\mathbf{P}$. For any $t \in \mathbf{R}$ let $F(t) := \mathbf{P}(X \leq t)$. For any $s \in (0,1)$ define

$$Y(s) := \sup\{t \in \mathbf{R}\colon F(t) < s\}.$$

Then Y is a random variable on $(0,1)$ with the uniform probability law on $(0,1)$. Show that X and Y are equal in distribution. That is, $\mathbf{P}(Y \leq t) = F(t)$ for all $t \in \mathbf{R}$.

Exercise 2 (Box-Muller Algorithm)**.** Let U_1, U_2 be independent random variables uniformly distributed in $(0,1)$. Define

$$R := \sqrt{-2\log U_1}, \qquad \Psi := 2\pi U_2.$$
$$X := R\cos\Psi, \qquad Y := R\sin\Psi.$$

Show that X, Y are independent standard Gaussian random variables. So, we can simulate any number of independent standard Gaussian random variables with this procedure.

Then, using Matlab or R (or another software package), verify that you can simulate these two Gaussian random variables by sampling from U_1, U_2. (For example, in Matlab, you can sample from these random variables using the `rand` function.) Using about 10^7 samples of U_1, U_2, plot a histogram of X and Y separately, and observe that it agrees with a histogram of a Gaussian.

Exercise 3. Let $Y_1, Y_2, \ldots$ be random variables such that $\sqrt{n}Y_n$ converges in distribution to a mean zero Gaussian random variable with variance $1/4$ as $n \to \infty$. Let

$$f(t) := \sin(t+2), \qquad \forall\, t \in \mathbf{R}$$

Show that, as $n \to \infty$, the random variables

$$\sqrt{n}(f(Y_n) - f(0))$$

converge in distribution to a random variable Z, and then compute $\mathbf{E}Z^4$.

Exercise 4. Let $Y_1, Y_2, \ldots$ be random variables such that $\sqrt{n}Y_n$ converges in distribution to a mean zero Gaussian random variable with variance $1/2$. Let

$$f(t) := \frac{e^t + e^{-t}}{2} = \cosh(t), \qquad \forall\, t \in \mathbf{R}$$

Show that, as $n \to \infty$, the random variables

$$n(f(Y_n) - f(0))$$

converge in distribution to a random variable Z, and then compute $\mathbf{E}Z$.

Exercise 5. Using a computer, create 10^4 independent samples $X_1,\ldots,X_n$ from a standard Gaussian distribution, and then observe on the computer that the sample mean and sample variance are uncorrelated. That is, plot the point $(\overline{X},S^2-1)$ in the plane, and plot such a point 10^3 times (each time independent of the others, plotting all 10^3 points simultaneously). Observe that these pairs of points demonstrate zero correlation.

Exercise 6. Using a computer, create $n=10^7$ independent samples $X_1,\ldots,X_n$ from a standard Gaussian distribution, and then create 10^7 independent samples $Y_1,\ldots,Y_n$ from a chi-squared distribution with 2 degrees of freedom, such that $Y_1,\ldots,Y_n,X_1,\ldots,X_n$ are all independent. Make a histogram of $X_1/\sqrt{Y_1/2},\ldots,X_n/\sqrt{Y_n/2}$. Plot also the density of Student's t-distribution with 2 degrees of freedom, to verify that the histogram agrees with this density.

(To get a nice picture, you might have to throw out values of the random variables $X_n/\sqrt{Y_n/2}$ that are large in absolute value.)