Quiz 4 occurs October 19, in the discussion section. The quiz will be based upon the problems below.

# Quiz 4 Problems

**Exercise 1.** Let $X_1,\ldots,X_n$ be a random sample of size $n=2$, so that $X_1$ is a sample from exponential distribution with unknown parameter $\theta>0$, so that $X_1$ has density $\theta e^{-x\theta}1_{x>0}$.

Suppose we want to estimate the mean

$$g(\theta):=1/\theta.$$

- Find the UMVU for $g(\theta)$. (Hint: what condition of equality is there for the Cramér-Rao inequality?)
- Show that $\sqrt{X_1X_2}$ has smaller mean squared error than the UMVU. That is,
$$\mathbf{E}(\sqrt{X_1X_2}-1/\theta)^2$$
is less than the variance of the UMVU.
- Does finding an estimator with smaller mean squared error contradict the definition of UMVU? Explain.
- (Optional) Find an estimator with even smaller mean squared error than $\sqrt{X_1X_2}$, for all $\theta\in\Theta$.

**Exercise 2.** Let $f\colon\mathbf{R}^n\to\mathbf{R}$ be a convex function. Let $x\in\mathbf{R}^n$ be a local minimum of $f$. Show that $x$ is in fact a global minimum of $f$.

Show also that if $f$ is strictly convex, then there is at most one global minimum of $f$.

Now suppose additionally that $f$ is a $C^1$ function (all derivatives of $f$ exist and are continuous), and $x\in\mathbf{R}^n$ satisfies $\nabla f(x)=0$. Show that $x$ is a global minimum of $f$.

**Exercise 3.** Let $A$ be a real $m\times n$ matrix. Let $x\in\mathbf{R}^n$ and let $b\in\mathbf{R}^m$. Show that the function $f\colon\mathbf{R}^n\to\mathbf{R}$ defined by $f(x)=\frac{1}{2}\left|\left|Ax-b\right|\right|^2$ is convex. Moreover, show that

$$\nabla f(x)=A^T(Ax-b),\qquad D^2f(x)=A^TA.$$

(Here $D^2f$ denotes the matrix of second derivatives of $f$.)

So, if $\nabla f(x)=0$, i.e. if $A^TAx=A^Tb$, then $x$ is the global minimum of $f$. And if $A$ has full rank, then $A^TA$ is invertible, so that $x=(A^TA)^{-1}A^Tb$ is the global minimum of $f$.

**Exercise 4.** Let $f_1,\ldots,f_n\colon\mathbf{R}\to\mathbf{R}$ be $n$ strictly convex functions on $\mathbf{R}$. Define $g\colon\mathbf{R}^n\to\mathbf{R}$ by

$$g(x_1,\ldots,x_n):=\sum_{i=1}^n f(x_i),\qquad\forall\,(x_1,\ldots,x_n)\in\mathbf{R}^n.$$

Show that $g\colon\mathbf{R}^n\to\mathbf{R}$ is strictly convex.

**Exercise 5.** Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be a $C^1$ function (all derivatives of $f$ exist and are continuous). Suppose there exists $z \in \mathbf{R}$ such that, for any $x_1 \in \mathbf{R}$, we have

$$f(x_1, z) < f(x_1, x_2), \qquad \forall\, x_2 \neq z.$$

Assume also that the function

$$x_1 \mapsto f(x_1, z)$$

is strictly convex. Show that $f$ has at most one global minimum.

**Exercise 6.** Let $X_1, \ldots, X_n$ be a random sample of size $n$ from a Poisson distribution with unknown parameter $\lambda > 0$. (So, $\mathbf{P}(X_1 = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$.)

- Find an MLE for $\lambda$.
- Find an MLE for $e^{-\lambda}$.
- How do your results compare to the previous homework, where we found two different estimators for $e^{-\lambda}$ (one from the method of moments, and the other by applying the Rao-Blackwell Theorem.)

**Exercise 7.** Let $X_1, \ldots, X_n$ be a random sample of size $n$ from a Gamma distribution with unknown $\alpha > 0$ and known $\beta > 0$

- Try to find an MLE of $\alpha$. (You might run into a difficulty in getting an explicit expression for $\alpha$.)
- Using a computer, after fixing some possible values of $X_1, \ldots, X_n$, find an MLE of $\alpha$ using any computational optimization method you want to use. Can you guarantee that you have found the global maximum?