Math 458, Fall 2022, USC — Instructor: Steven Heilman

Name: ______________ USC ID: ______________ Date: ______________

Signature: ______________. Discussion Section: ______________
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 2

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck![a]

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample or explain your reasoning**.

   (a) (2 points) The smallest positive number that exists in double precision floating point arithmetic is

   $$2^{-1022}.$$

   TRUE FALSE (circle one)

   (b) (2 points) In double precision floating point arithmetic, the closest number to 4 that is larger than 4 is

   $$4 + 2^{-52}.$$

   (Put another way, if $x > 4$ is a double precision floating point number, then $|x - 4|$ is minimized when $x = 4 + 2^{-52}$.)

   TRUE FALSE (circle one)

   (c) (2 points) QR Decompositions are unique. That is, if A is an $n \times n$ real matrix, then there is exactly one $n \times n$ orthogonal matrix Q and there is exactly one $n \times n$ upper triangular matrix R such that A can be written as

   $$A = QR.$$

   TRUE FALSE (circle one)

(d) (2 points) Let $a_0, \ldots, a_{10}, b_0, \ldots, b_{10}$ be real numbers. Then there is a unique polynomial p of degree at most 10 such that

$$p(a_i) = b_i \qquad \forall\, 0 \le i \le 10.$$

TRUE FALSE (circle one)

(e) (2 points) Let $m, n \ge 1$ be integers. Any real $m \times n$ matrix A can be written as

$$A = UDV$$

where U is an orthogonal $m \times m$ matrix, V is an orthogonal $n \times n$ matrix, and D is an $m \times n$ matrix whose non-diagonal entries are zero (i.e. $D_{ij} = 0$ whenever $1 \le i \le m, 1 \le j \le n$ and $i \ne j$.)

TRUE FALSE (circle one)

2. (10 points) Let $n \geq 1$ be an integer. Suppose I have a function $f\colon \mathbf{R} \to \mathbf{R}$ and I want to choose a polynomial p_n that interpolates f on the interval $[-1,1]$. That is, we would like to choose nodes $a_0, \ldots, a_n \in [-1,1]$ such that

$$f(a_i) = p_n(a_i), \qquad \forall\, 0 \leq i \leq n. \qquad (\ddagger)$$

- Suppose we want to choose the nodes $a_0, \ldots, a_n$ such that $\max_{y \in [-1,1]} |f(y) - p_n(y)|$ is as small as possible. Which nodes $a_0, \ldots, a_n$ would you choose? Justify your answer as best you can.

- Suppose $n = 2$ and $a_0 = 0$, $a_1 = 1$ and $a_2 = 2$. Suppose also that

$$f(x) = x^3, \qquad \forall\, x \in \mathbf{R}.$$

Write an explicit formula for the degree 2 polynomial p_2 satisfying $(\ddagger)$. Simplify your answer to the best of your ability.

3. (10 points) Suppose we have data points $(-1, 1), (0, 3), (1, 3) \in \mathbf{R}^2$ denoted as $\{(a_i, b_i)\}_{i=1}^3$. Find the line that best fits the data. That is, find the line $f\colon \mathbf{R} \to \mathbf{R}$ that minimizes the sum of squared differences $\sum_{j=1}^3 |f(a_i) - b_i|^2$.

   (You should find an exact formula for f. Do not write a Matlab program in this question.)

4. (10 points) Let A be a real $n \times n$ symmetric positive definite matrix all of whose eigenvalues are distinct.

   Write a Matlab program that applies the QR algorithm to A. The output of the program should be two real $n \times n$ matrices D and Q, such that D is a diagonal matrix containing the eigenvalues of D, and Q is an orthogonal matrix whose columns are eigenvectors of A, so that $A = QDQ^T$. (You are allowed to use the built-in Matlab program `qr`, whose syntax is `[Q,R]=qr(A)`, outputting a factorization $A = QR$.)

   (It is okay if QDQ^T is only approximately equal to A and D is only approximately diagonal, so that all non-diagonal entries of D are small.)

   In this problem, you will be graded on writing correct syntax in Matlab. Syntax mistakes will result in deductions of points.

5. (10 points) Let $n = 500$. Let A be a real $n \times n$ symmetric positive definite matrix all of whose eigenvalues are distinct. Suppose the largest eigenvalue of A is 1 and all other eigenvalues of A are at most $1/2$.

   Suppose $x \in \mathbf{R}^n$ is a nonzero vector such that $Ax = x$.

   Describe, to the best of your ability, the matrix

$$A^{1000}.$$

   Justify your answer. Simplify your answer as best you can.

   You should be able to say what each row of A^{1000} is, within a reasonably small margin of error. For example, you should have a fairly precise description of the 356^{th} row of A.

(Scratch paper)