Math 458, Fall 2022, USC Instructor: Steven Heilman

Name: ____________________ USC ID: ______________ Date: ____________

Signature: ____________________. Discussion Section: ____________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

# Final Exam, Part 1

This exam contains 8 pages (including this cover page) and 5 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 60 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Do not write in the table to the right. Good luck![a]

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 50 | |

1. Label the following statements as TRUE or FALSE. If the statement is true, **explain your reasoning**. If the statement is false, **provide a counterexample or explain your reasoning**.

   (a) (2 points) Let $A$ be an $n \times n$ real matrix with rank $n$. Let $b \in \mathbf{R}^n$. It is possible to solve the equation $Ax = b$ for an unknown $x \in \mathbf{R}^n$ using at most $1000n^5$ arithmetic operations.

   TRUE FALSE (circle one)

   (b) (2 points) Let $A$ be an $n \times n$ real matrix. It is possible to compute the determinant of $A$ using at most $1000n^5$ arithmetic operations.

   TRUE FALSE (circle one)

   (c) (2 points) If $A$ is a real $2 \times 2$ symmetric positive definite matrix, then there exist $2 \times 2$ matrices $L, U$ such that $L$ is lower triangular, $U$ is upper triangular, and such that $A$ can be written as

$$A = LU.$$

   TRUE FALSE (circle one)

(d) (2 points) Suppose we have data points $(-1,1),(0,3),(1,3) \in \mathbf{R}^2$ denoted as $\{(a_i,b_i)\}_{i=1}^{3}$. Then there exists a unique degree 4 polynomial $f\colon \mathbf{R} \to \mathbf{R}$ such that $f(a_i) = b_i$ for all $1 \leq i \leq 3$.

TRUE FALSE (circle one)

(e) (2 points) Let $A$ be a real $n \times n$ symmetric positive definite matrix. Suppose the largest eigenvalue of $A$ is 1, the second largest eigenvalue of $A$ is $1/2$, and all other eigenvalues of $A$ are at most $1/4$. Suppose $x \in \mathbf{R}^n$ is a nonzero vector such that $Ax = x$. Then

$$\left|\left|A - xx^T\right|\right|_{2\to 2} = 1/2.$$

(Recall that $||A||_{2\to 2} = \max_{x\in\mathbf{R}^n\colon ||x||\leq 1} ||Ax||$, and $||x|| = (x_1^2 + \cdots + x_n^2)^{1/2}$ for all $x = (x_1,\ldots,x_n) \in \mathbf{R}^n$.)

TRUE FALSE (circle one)

2. (a) (5 points) Describe the output of the following Matlab program. (Note that `Inf` represents infinity in Matlab so `2<Inf` evaluates to `1` and `Inf<Inf` evaluates to `0`.) As usual, justify your answer.

```
x=2;
n=1;
while x<Inf
    x=2^x;
    n=n+1;
end
n
```

(b) (5 points) Consider the initial value problem

$$\begin{cases} y'(t) = (y(t))^2 + 3, & \forall\, t > 0, \\ y(0) = 0 \end{cases}$$

Write a Matlab program that outputs an estimate for $y(1/10)$, using first order Euler's method with a step size of $h = 1/100$.

Your program will be graded on syntax and correctness.

3. (a) (5 points) Consider the differential equation

$$\begin{cases} y''(t) + 3y'(t) - 2y(t) = 0, & \forall\, t > 0, \\ y(0) = 2 \\ y'(0) = -3. \end{cases}$$

Find a matrix $A$ and a vector $b$ such that this differential equation can be equivalently written as an initial value problem of the form

$$\begin{cases} \frac{d}{dt}u(t) = Au(t), & \forall\, t > 0, \\ u(0) = b \end{cases}$$

As usual, justify your answer.

(b) (5 points) Recall that $\cos(t)$ and $\sin(t)$ are functions of $t$ satisfying $y''(t) = -y(t)$ for all $t \in \mathbf{R}$. In theory, we can then solve a boundary value problem explicitly using a sum of these functions.

Write an explicit solution of the boundary value problem

$$\begin{cases} y''(t) = -y(t), & \forall\, t \in [0, \pi/2], \\ y(0) = 1 \\ y(\pi/2) = 5. \end{cases}$$

4. (10 points) Consider the following initial value problem for $y\colon [0,\infty) \to (0,\infty)$.

$$\begin{cases} y'(t) = -5y(t), & \forall\, t > 0, \\ y(0) = 3. \end{cases}$$

- Prove that there exists a unique $y\colon [0,\infty) \to (0,\infty)$ solving this initial value problem such that $y'$ exists and is continuous.

  (Hint: you can freely use that if $g\colon (0,\infty) \to \mathbf{R}$ is a continuous function such that $g'(t)$ is constant for all $t > 0$, then $g$ is a linear function.)
- What function $y(t)$ solves this initial value problem? Justify your answer.

5. (10 points) Suppose $w(x) := 1$ for all $x \in [-1, 1]$.

- Find constants $b, c \in \mathbf{R}$ such that the polynomial

$$q(x) = x^2 + bx + c$$

is $w$-orthogonal to all polynomials of degree at most 1. That is, if $f$ is a polynomial of degree at most 1, we have

$$\int_{-1}^{1} f(x)q(x)w(x)dx = 0.$$

- Suppose

$$g(x) = 3\sqrt{3}x^3 - 6x^2 - \sqrt{3}x + 5, \qquad \forall\, x \in [-1, 1].$$

Compute

$$\int_{-1}^{1} g(x)w(x)dx$$

using the Gaussian quadrature rule

$$\int_{-1}^{1} g(x)w(x)dx = \sum_{i=0}^{n} b_i g(a_i).$$

(Hint: recall that $n = 1$ and $b_i = \int_{-1}^{1} w(x) \prod_{j\in\{0,\ldots,n\}:\, j\neq i} \frac{x-a_j}{a_i-a_j} dx$.)

(Scratch paper)