Please provide complete and well-written solutions to the following exercises.

Due September 8, 10AM PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

Homework 3

Exercise 1 (Numerical Integration)**.** Consider the function

$$f(t) := t^3 + 1.$$

In this case, we can easily compute

$$\int_0^1 f(t)dt = \frac{5}{4}.$$

Sometimes, especially in computer graphics applications, integrals are too complicated to compute directly, so we instead use randomness to estimate the integral. That is, we pick n random points in $[0, 1]$, and average the values of f at these points, as in the following program.

```
n=10^5;
f= @(t) t.^3 +1;
mean(f(rand(1,n)))
```

Using this program with $n = 10^5, 10^6, 10^7$ and 10^8, report the estimated values for the integral of f, along with their relative errors.

Now, compute the exact value of $\int_3^5 \log(x)\, dx$, and modify the above program to give estimates for the value of this integral and report relative errors, using a number of points n where $n = 10^5, 10^6, 10^7$ and 10^8.

Exercise 2. Using the Divide and Conquer Algorithm to approximate the value of $\sqrt{3}$ with the function $f(x) := x^2 - 3$ by starting your search on the interval $[1, 2]$. Report how many iterations the algorithm takes until it no longer makes any progress (i.e. once the algorithm can no longer create a smaller interval to search for a zero of f.)

Then, use the Babylonian square root algorithm to approximate the value of $\sqrt{3}$, starting with $a_1 = 1$. Report how many iterations the algorithm takes until it no longer makes any progress (i.e. once the sequence generated by the algorithm becomes constant.)

Exercise 3. Suppose we use Newton's Method for the function $f(t) := t^2 - 2$ with initial guess $x_1 = 2$, i.e. we use the Babylonian Square Root Algorithm. (We know that $f(\sqrt{2}) = 0$.) In this Exercise, we will prove that $x_1, x_2, \ldots$ converges to $\sqrt{2}$ using the following strategy.

- Show that $x_k \geq \sqrt{2}$ implies that $x_{k+1} \geq \sqrt{2}$ for all $k \geq 1$. Since $x_1 > \sqrt{2}$, conclude that $x_k \geq \sqrt{2}$ for all $k \geq 1$. (If $g(t) := t/2 + 1/t$, show that $g(t) > \sqrt{2}$ for all $t > \sqrt{2}$ by considering $g(\sqrt{2})$ and showing that $g'(t) > 0$ for all $t \geq \sqrt{2}$, and then apply the Fundamental Theorem of Calculus.)
- Show that $x_1 \geq x_2 \geq x_3 \geq \cdots$. That is, the sequence $x_1, x_2, \ldots$ is decreasing.
- You can freely use the following fact from real analysis: if a sequence $x_1, x_2, \ldots$ of real numbers is decreasing with $x_k \geq \sqrt{2}$ for all $k \geq 1$, then the sequence converges to a real number $x \geq \sqrt{2}$. That is, there exists $x \geq \sqrt{2}$ such that $x = \lim_{k\to\infty} x_k$.
- Define $\phi(t) := t - \frac{f(t)}{f'(t)}$ for any $t \geq \sqrt{2}$. Note that ϕ is continuous, so $\phi(x) = \lim_{k\to\infty} \phi(x_k)$. Using $\phi(x_k) = x_{k+1}$, conclude that $\phi(x) = x$. Using $\phi(x) = x$, deduce that $x = \sqrt{2}$.

Exercise 4. Consider the function

$$f(x) := \mathrm{sign}(x)\sqrt{|x|}, \qquad \forall\, x \in \mathbf{R}.$$

This function has only one zero at $x = 0$.

Either by hand or a computer, use Newton's method to try to find a zero, using an initial guess such as $1, 1/2$ or $1/4$.

What happens? Describe the output of Newton's Method. Does the sequence of iterates $x_0, x_1, \ldots$ converge to 0? Explain why or why not. (Does the convergence theorem for Newton's Method apply?) (You can look up any convergence theorem for Newton's method from the notes or book to answer this question.)

Exercise 5. Consider the function

$$f(x) := x^{1/3}, \qquad \forall\, x \in \mathbf{R}.$$

This function has only one zero at $x = 0$.

Using a computer, use Newton's method to try to find a zero, using an initial guess such as $1, 1/2$ or $1/4$.

What happens? Describe the output of Newton's Method. Does the sequence of iterates $x_0, x_1, \ldots$ converge to 0? Explain why or why not. (Does the convergence theorem for Newton's Method apply?) (You can look up any convergence theorem for Newton's method from the notes or book to answer this question.)

Exercise 6. Consider the function

$$f(x) := x^2 + 1/100, \qquad \forall\, x \in \mathbf{R}.$$

This function has no zeros, but it is nearly zero at $x = 0$. For instructive purposes, we can still apply Newton's Method to see what happens.

Using a computer, use Newton's method to try to find a zero of f, using an initial guess of $1/10$.

What happens? Describe the output of Newton's Method. Explain what has happened.