Please provide complete and well-written solutions to the following exercises.

Due November 10, 1159PM PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

Homework 9

Exercise 1. For any $x \in (-1, 1)$, let $w(x) := 1/\sqrt{1 - x^2}$. Let P_n be the set of polynomials of degree at most n on $[-1, 1]$.

- Show that the Chebyshev polynomials $T_0, \ldots, T_n$ are a w-orthogonal basis of P_n.
- Give an explicit formula for the nodes of the Gaussian quadrature that uses $n + 1$ nodes and this w.

Exercise 2. This exercise investigates Matlab's standard routine for evaluating double integrals. We are particularly interested in computational time. Suppose we integrate the function

$$f(x, y) = \sqrt{1 - (x^2 + y^2)}$$

over the unit disc $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$ for decreasing tolerances with the following program.

```
fprintf('    tol       estimated Q      relerror   Computation Time (s)\n')
fprintf('************************************************\n');
for k=1:12
    tol=10^(-k);             %selected tolerance
    tic;                        %this command starts a timer
    Q=dblquad(@(x,y) (sqrt(1-(x.^2+y.^2))).*(x.^2+y.^2<=1),-1,1,-1,1,tol);
    comptime=toc;             % 'toc' records the time elapsed since 'tic'
    actual = (2/3)*pi;        % actual value of the integral
    relerror=abs(Q-actual)/actual;
    fprintf('%8.0e %21.14f %3.2e %7.3f\n', ...
        tol,Q,relerror,comptime);
end
```

- How does the computation time vary with the tolerance?
- Now, modify this program to integrate the function

  $$g(x, y) = 1 + \sqrt{1 - (x^2 + y^2)}.$$

  over the unit disc $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$. How does the computation time vary with the tolerance? If you observe long computation times, try to explain why they have occurred.

Exercise 3. Estimate the integral

$$\int_0^1 \frac{\cos(x)}{\sqrt{x - x^2}} dx.$$

Use whatever built in Matlab functions you want to use (more is perhaps better than less). Do your best to justify what the correct answer is.

(Hint: consider making the substitution $s = 2x - 1$ and then $s = \cos\theta$.)

Exercise 4. Consider the following initial value problem.

$$\begin{cases} y''(t) = -y(t) & \forall\, t \geq 0 \\ y'(0) = 1 \\ y(0) = 0 \end{cases}.$$

Verify that $y(t) = \sin(t)$ solves this problem.

We can write the differential equation in our usual form by defining $z(t) := \begin{pmatrix} y'(t) \\ y(t) \end{pmatrix} \in \mathbf{R}^2$ for all $t \geq 0$. We can then rewrite the initial value problem as

$$z'(t) = Az(t), \qquad \forall\, t \geq 0, \qquad z(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Use the approximation $z'(t) \approx [z(t+h) - z(t)]/h$ to approximately solve the differential equation in Matlab, i.e. use Euler's method to write

$$z(t+h) - z(t) \approx hAz(t).$$

In this approximate solution, use $h = 10^{-k}$ for $k = 1, \ldots, 5$, and compute the relative error of $y((100 + 1/2)\pi)$ as a function of h.

Exercise 5. Consider the following initial value problem

$$y'(t) = \sqrt{|t|}, \qquad \forall\, t \in [-2, 2], \qquad y(-2) = -1.$$

- Verify that there is a solution to this initial value problem of the form
  $$f(t) = \begin{cases} \frac{2\sqrt{(-t)}\, t}{3} + c & , \text{if}\, t < 0 \\ \frac{2t^{3/2}}{3} + c & , \text{if } t \geq 0. \end{cases}$$
  Moreover, find the correct value of c.
- Using Matlab, compute a numerical solution of this initial value problem with Euler's method and different step sizes h, on the interval $[-2, 2]$.
- Should you be concerned with your results from Euler's method? For what value of t is the absolute error the highest? Naïvely, one might measure the error of Euler's method, by just examining the final endpoint of the interval $y(2)$, and comparing the computed value with the exact value. Is this sensible in this example?

Exercise 6. This exercise begins by investigating some solutions of the Lorentz equations which are defined for $y\colon \mathbf{R} \to \mathbf{R}^3$ by

$$y'(t) = A(t)y(t)$$

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} \qquad \text{where} \qquad A(t) = \begin{pmatrix} -\beta & 0 & y_2(t) \\ 0 & -\sigma & \sigma \\ -y_2(t) & \rho & -1. \end{pmatrix}$$

Here $\sigma = 10$, $\beta = \frac{8}{3}$, and we vary the value of ρ. (Since A depends on $y_2(t)$, the differential equation $y' = Ay$ is nonlinear, leading to some interesting behavior.) Solutions of this system display various periodicities that give the appearance of a particle orbiting around two different points. These two points are called attractors. Since the behavior of the solution seems rather unpredictable, the solution y is called chaotic. In our investigation, the initial value of y is started near one of the two attractors, i.e. with $\eta = +\sqrt{\beta(\rho - 1)}$ there is an attractor at the point $r_1 = (\rho - 1, \eta, \eta) \in \mathbf{R}^3$ and we start at the initial value $r_1 + (0, 0, 3) \in \mathbf{R}^3$.

- In varying our parameter ρ, using the program `lorenzgui`, check the values of ρ which produce non-chaotic orbits. (You should check five different ρ values, as specified in `lorenzgui`.) Then, we label the periodicity of the orbits with a series of pluses and minuses, where a plus denotes a circuit around one attractor and a minus denotes a circuit around the other attractor.

  For example, if the solution y goes around attractor one, then attractor two, then one, then two, etc. label this periodicity as $+-$.
- Run the Matlab demo function `orbitode` which demonstrates the use of events in an ODE solver. This solution of this problem gives the orbit of a small body around two larger bodies. (Think for example about a spacecraft's orbit under the influence of the Earth and moon's gravity.) What roles to the variables `te`, `ye`, `y`, `ie` play in the program?
- Mimic the program `orbitode` and implement an event function in the Lorenz problem. Using this event function, find the periods of the periodic orbits which occur for different ρ values. That is, for a given ρ value, find the smallest nonzero time T such that $y(t + T) = y(t)$. (This equality will probably not hold exactly, so you should probably check for the smallest nonzero time T such that $|y(t + T) - y(t)| < 10^{-6}$ for all larger t, or using some other small number other than 10^{-6}).