505B Midterm 1 Solutions[1]

1. Question 1

Consider a standard 8×8 chess board. Let V be a set of vertices corresponding to each square on the board (so V has 64 elements). Any two vertices $x, y \in V$ are connected by an edge if and only if a king can move from x to y. (The king chess piece can move either one space along the horizontal, one space along the vertical axis, or one space diagonally.) Consider the simple random walk on this graph. This Markov chain then represents a king randomly moving around a chess board. For every space x on the chessboard, compute the expected return time $\mathbf{E}_x T_x$ for that space. (It might be convenient to just draw the expected values on the chessboard itself.)

Solution. By inspection, the Markov chain is irreducible (with positive probability, the king can move from any space on the board to any other space on the board in at most seven steps). Since the Markov chain is finite and irreducible, Theorems 3.34 and 3.37 say that the stationary distribution π exists and is unique. By Corollary 3.38, if π is the unique solution to $\pi = \pi P$, then $\mathbf{E}_x T_x = 1/\pi(x)$. So, it suffices to find $\pi(x)$ for any $x \in V$. From Example 3.51 in the notes, $\pi(x) = \deg(x)/\sum_{y \in V} \deg(y)$ for all $x \in V$. The following table depicts the degrees of each entry in the chess board

$$\begin{pmatrix} 3 & 5 & 5 & 5 & 5 & 5 & 5 & 3 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 3 & 5 & 5 & 5 & 5 & 5 & 5 & 3 \end{pmatrix}.$$

So, $\sum_{x \in V} \deg(x) = 8 \cdot 36 + 5 \cdot 24 + 3 \cdot 4 = 420$, so $\mathbf{E}_x T_x = 1/\pi(x) = 420/\deg(x)$. So, the following table depicts $\mathbf{E}_x T_x$ at each point on the chessboard.

$$\begin{pmatrix} 140 & 84 & 84 & 84 & 84 & 84 & 84 & 140 \\ 84 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 84 \\ 84 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 84 \\ 84 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 84 \\ 84 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 84 \\ 84 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 84 \\ 84 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 52.5 & 84 \\ 140 & 84 & 84 & 84 & 84 & 84 & 84 & 140 \end{pmatrix}.$$

2. Question 2

Let Ω be a finite state space. This problem demonstrates that the total variation distance is a metric. That is, show that the following three properties are satisfied:

- $\|\mu - \nu\|_{\mathrm{TV}} \geq 0$ for all probability distributions μ, ν on Ω, and $\|\mu - \nu\|_{\mathrm{TV}} = 0$ if and only if $\mu = \nu$.
- $\|\mu - \nu\|_{\mathrm{TV}} = \|\nu - \mu\|_{\mathrm{TV}}$

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- $\|\mu - \nu\|_{\text{TV}} \leq \|\mu - \eta\|_{\text{TV}} + \|\eta - \nu\|_{\text{TV}}$ for all probability distributions μ, ν, η on Ω.

(Hint: you may want to use the triangle inequality for real numbers: $|x - y| \leq |x - z| + |z - y|$, $\forall\ x, y, z \in \mathbf{R}$.)

Solution. By its definition, $\|\mu - \nu\|_{\text{TV}} \geq 0$ is always true, and if $\mu = \nu$ then $\|\mu - \nu\|_{\text{TV}} = 0$. Since $\|\mu - \nu\|_{\text{TV}} = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)|$, if $\|\mu - \nu\|_{\text{TV}} = 0$, then $\mu(x) = \nu(x)$ for all $x \in V$, i.e. $\mu = \nu$. For the second, property, note that

$$\|\mu - \nu\|_{\text{TV}} = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)| = \frac{1}{2}\sum_{x\in\Omega} |\nu(x) - \mu(x)| = \|\nu - \mu\|_{\text{TV}}\,,$$

For the final property, we use the triangle inequality for real numbers to get

$$\begin{aligned}\|\mu - \nu\|_{\text{TV}} &= \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)| = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \eta(x) + \eta(x) - \nu(x)| \\ &\leq \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \eta(x)| + \frac{1}{2}\sum_{x\in\Omega} |\eta(x) - \nu(x)| = \|\mu - \eta\|_{\text{TV}} + \|\eta - \nu\|_{\text{TV}}\,.\end{aligned}$$

3. Question 3

Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1, 2, 3\}$ and with the following transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

- Is the Markov chain irreducible? Prove your assertion.
- Classify all states in the Markov chain as recurrent or transient.
- Is the Markov chain aperiodic? Prove your assertion.
- List all eigenvalues of the matrix P.
- Find a stationary distribution π, if it exists. (If it does not exist, prove it.) Is the stationary distribution unique?

Solution. Evidently $\max(P(i, j), P^2(i, j), P^3(i, j)) = 1$ for all $i, j \in \Omega$. Therefore, the Markov chain is irreducible, and $\mathbf{P}_i(T_i < \infty) = \mathbf{P}_i(T_i \leq 3) = 1$ for all $i \in \Omega$, so all states in the chain are recurrent. For any $i \in \Omega$, $\mathcal{N}(i) = \{3, 6, 9, \ldots\}$, so $\gcd\mathcal{N}(i) = 3$ for all $i \in \Omega$, i.e. the chain is not aperiodic (all states have period 3). An eigenvalue $\lambda \in \mathbf{C}$ satisfies $\det(P - \lambda I) = 0$, i.e.

$$0 = \det(P - \lambda I) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{pmatrix} = -\lambda^3 + 1.$$

So, $\lambda^3 = 1$, i.e. all third roots of unity are eigenvalues of P. That is, the eigenvalues are

$$\{1, -(1/2) + \sqrt{-3}/2, -(1/2) - \sqrt{-3}/2\}.$$

Since the Markov chain is finite and irreducible, Theorems 3.34 and 3.37 say that the stationary distribution exists and is unique. By inspection, $\pi = (1/3, 1/3, 1/3)$ satisfies $\pi = \pi P$.

4. Question 4

(In this problem, you are allowed to use a computer or calculator for the purpose of computing powers of the matrix P.)

Suppose we have a Markov Chain $(X_0, X_1, \ldots)$ with state space $\Omega = \{1,2,3,4,5,6\}$ and with the following transition matrix

$$P = \begin{pmatrix} 0 & 0 & .1 & .2 & .4 & .3 \\ 0 & 0 & .1 & .2 & .4 & .3 \\ .3 & .4 & .3 & 0 & 0 & 0 \\ .1 & .2 & .4 & .3 & 0 & 0 \\ 0 & .1 & .2 & .4 & .3 & 0 \\ 0 & 0 & .1 & .2 & .4 & .3 \end{pmatrix}.$$

- Is the Markov chain irreducible? Prove your assertion.
- Classify all states in the Markov chain as recurrent or transient.
- Is the Markov chain aperiodic? Prove your assertion.
- Find a stationary distribution π, if it exists. (If it does not exist, prove it.) (If it does exist, you can just report each entry of π to four decimal places of accuracy.) Is the stationary distribution unique?

Solution. Using e.g. Matlab, we observe that $P^2(i,j) > 0$ for all $i,j \in \Omega$. Therefore, the Markov chain is irreducible, and Lemma 3.28 (by letting $k \to \infty$) implies that $\mathbf{P}_i(T_i < \infty) = 1$ for all $i \in \Omega$, so all states in the chain are recurrent. Since $P^2(i,j) > 0$ for all $i,j \in \Omega$, we have $P^m(i,j) > 0$ for all $i,j \in \Omega$ and for all $m \geq 2$. So, e.g. $\mathcal{N}(1) \supseteq \{2,3\}$, so that $\gcd \mathcal{N}(1) = 1$. From Lemma 3.57, all states in a finite irreducible Markov chain have the same period. So, all states of this Markov chain have period 1, i.e. the chain is aperiodic. Since the Markov chain is finite and irreducible, Theorems 3.34 and 3.37 say that the stationary distribution exists and is unique. To find the stationary distribution π, we take the matrix P to a high power using e.g. Matlab. The Convergence Theorem implies that P^n converges to a matrix all of whose rows are π as $n \to \infty$. Evidently $n = 15$ is enough to find four decimal places of accuracy, obtaining

$$\pi \approx (.0909, .1556, .2310, .2156, .2012, .1056).$$