Graduate Applied Probability II, Spring 2021, USC Instructor: Steven Heilman

Name: ____________________ USC ID: ______________ Date: ____________

Signature: ____________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Exam 2

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 24 hours to complete the exam.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right. Good luck![a]

1. (10 points) Let $S_0, S_1, \ldots$ denote the simple random walk on $\mathbf{Z}$ with $S_0 := 0$. Let n and c be positive integers. Show that:

$$\mathbf{P}\left(\max_{1\leq j\leq n} |S_j| \geq c\right) \leq 2\mathbf{P}(|S_n| \geq c).$$

(Hint: is this related at all to the reflection principle?)

2. (10 points) Let $(X_0, X_1, \ldots)$ be the simple random walk on $\mathbf{Z}$. For any $n \geq 0$, define $M_n = X_n^3 - 3nX_n$. Show that $(M_0, M_1, \ldots)$ is a martingale with respect to $(X_0, X_1, \ldots)$

   Now, fix $m > 0$ and let T be the first time that the walk hits either 0 or m. Show that, for any $0 < k \leq m$,

$$\mathbf{E}_k(T \mid X_T = m) = \frac{m^2 - k^2}{3}.$$

   (You are allowed to apply the Optional Stopping Theorem Version 2 without verifying boundedness of the martingale.)

3. (10 points) Let $m \geq 1$ be an integer. Let P be the $m \times m$ transition matrix of a finite (discrete-time) reversible, irreducible Markov chain. Denote the eigenvalues of P as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. (You can take it as given that the eigenvalues are real, since we verified this in an exercise.) Show:

   - $\lambda_i \leq 1$ for all $1 \leq i \leq m$.
   - $|\lambda_i| \leq 1$ for all $1 \leq i \leq m$.
   - $\lambda_1 > \lambda_2$.
   - If additionally P is aperiodic, then $\lambda_m > -1$.

4. (10 points) Let Ω be a finite set. Let π be a probability distribution on Ω. For any $0 < p < \infty$, and for any $f \in \mathbf{R}^{\Omega}$, define

$$||f||_{p,\pi} := \Big(\sum_{x\in\Omega} |f(x)|^{p}\,\pi(x)\Big)^{1/p}, \qquad \mathbf{E}_{\pi}f := \sum_{x\in\Omega} f(x)\pi(x).$$

- Show that $||f||_{1,\pi} \leq ||f||_{2,\pi}$ for any $f \in \mathbf{R}^{\Omega}$.
- Let μ be a probability distribution on Ω. If $\pi(x) > 0$ for all $x \in \Omega$, show that

$$||\mu - \pi||_{\mathrm{TV}} \leq \frac{1}{2}\left|\left|\frac{\mu(\cdot)}{\pi(\cdot)} - 1\right|\right|_{2,\pi}.$$

- Let $t > 0$. Prove the bound

$$\max_{x\in\Omega}\left|\left|\frac{H_t(x,\cdot)}{\pi(\cdot)} - 1\right|\right|_{2,\pi} \leq \sup_{f\in\mathbf{R}^{\Omega}\colon\, ||f||_{2,\pi}\leq 1}\ \max_{x\in\Omega}\left|[(H_t - \mathbf{E}_{\pi})f](x)\right|.$$

  (Hint: show, for any $f \in \mathbf{R}^{\Omega}$, $||f||_{2,\pi} = \sup_{g\in\mathbf{R}^{\Omega}\colon\, ||g||_{2,\pi}\leq 1} |\langle f, g\rangle_{\pi}|$.)

Let P be the transition matrix of a finite, irreducible (discrete-time) Markov chain, let π be the unique stationary distribution of the chain, and let $H_t := e^{t(P-I)}$, $t \geq 0$, be the corresponding heat kernel. The mixing time of a continuous-time Markov chain measures how rapidly the Markov chain converges to equilibrium, providing a more quantitative estimate than the convergence theorem provides. For any $\varepsilon > 0$, we define

$$t_{\mathrm{mix}}(\varepsilon) := \inf\Big\{t \geq 0\colon\ \max_{x\in\Omega} ||H_t(x,\cdot) - \pi(\cdot)||_{\mathrm{TV}} \leq \varepsilon\Big\}.$$

The mixing time of the continuous-time Markov chain is defined to be $t_{\mathrm{mix}}(1/4)$.

- Suppose we have a finite, irreducible, reversible Markov chain with spectral gap $\gamma := 1 - \lambda_2$. Prove that $t_{\mathrm{mix}} \leq \frac{1}{\gamma}\log(2/\sqrt{\min_{y\in\Omega}\pi(y)})$ by first proving

$$\max_{x\in\Omega} ||H_t(x,\cdot) - \pi(\cdot)||_{\mathrm{TV}} \leq \frac{1}{2}\frac{e^{-\gamma t}}{\sqrt{\min_{y\in\Omega}\pi(y)}}, \qquad \forall\, t > 0$$

  (Hint: if $g_1, \ldots, g_{|\Omega|} \in \mathbf{R}^{\Omega}$ are an orthonormal basis of eigenfunctions of P, and if $x \in \Omega$ and $e_x \in \mathbf{R}^{\Omega}$ satisfies $e_x(x) := 1$ and $e_x(y) := 0$ for all $y \neq x$, then show that $\pi(x) = \langle e_x, e_x\rangle_{\pi} = (\pi(x))^2 \sum_{j=1}^{|\Omega|} |g_j(x)|^2$.)

5. (10 points) Let n be a positive integer and denote $i := \sqrt{-1}$. Let $\tau := e^{2\pi i/n}$. Let $\Omega := \{\tau, \tau^2, \tau^3, \ldots, \tau^{n-1}, 1\}$ be the set of n^{th} roots of unity. Let P be the transition matrix such that $P(\omega, \tau\omega) = P(\omega, \tau^{-1}\omega) = 1/2$ for all $\omega \in \Omega$. That is, P is the simple random walk on the cyclic group of n elements.

- Show that the (discrete time) Markov chain with transition matrix P is reversible with stationary distribution $\pi(x) := 1/n$ for all $x \in \Omega$.
- Show that the eigenvalues of P are $\{\cos(2\pi j/n)\}_{j=0}^{n-1}$. Consequently, the spectral gap is $\gamma := 1 - \cos(2\pi/n)$.
- Bound the mixing time of the continuous-time Markov chain corresponding to P, using your result from the previous problem.

(Scratch paper)

(Extra Scratch paper)