Please provide complete and well-written solutions to the following exercises.

Due February 12, 12PM noon PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

Homework 2

Exercise 1. Let P, Q be stochastic matrices of the same size. Show that PQ is a stochastic matrix. Conclude that, if r is a positive integer, then P^r is a stochastic matrix.

Exercise 2. Let A, B be events in a sample space. Let $C_1, \ldots, C_n$ be events such that $C_i \cap C_j = \emptyset$ for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and such that $\cup_{i=1}^n C_i$ is the whole sample space. Show:

$$\mathbf{P}(A|B) = \sum_{i=1}^{n} \mathbf{P}(A|B,\, C_i)\mathbf{P}(C_i|B).$$

Exercise 3. Let $0 < p, q < 1$. Let $P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$. Find the (left) eigenvectors of P, and find the eigenvalues of P. By writing any row vector $x \in \mathbf{R}^2$ as a linear combination of eigenvectors of P (whenever possible), find an expression for xP^n for any $n \geq 1$. What is $\lim_{n\to\infty} xP^n$? Is it related to the vector $\pi = (q/(p+q), p/(p+q))$?

Exercise 4. Let $G = (V, E)$ be a graph. Let $|E|$ denote the number of elements in the set E, i.e. $|E|$ is the number of edges of the graph. Prove: $\sum_{x\in V} \deg(x) = 2\,|E|$.

Exercise 5. Let M, N be stopping times for a Markov chain $X_0, X_1, \ldots$. Show that $\max(M, N)$ and $\min(M, N)$ are stopping times. In particular, if $n \geq 0$ is fixed, then $\max(M, n)$ and $\min(M, n)$ are stopping times

Exercise 6. Let A, B be events such that $B \subseteq \{X_0 = x_0\}$. Then $\mathbf{P}(A|B) = \mathbf{P}_{x_0}(A|B)$.

More generally, if A, B are events, then $\mathbf{P}_{x_0}(A|B) = \mathbf{P}(A|B, X_0 = x_0)$.

Exercise 7. Suppose we have a Markov Chain with state space Ω. Let $n \geq 0$, $\ell \geq 1$, let $x_0, \ldots, x_n \in \Omega$ and let $A \subseteq \Omega^\ell$. Using the (usual) Markov property, show that

$$\begin{aligned}\mathbf{P}((X_{n+1}, \ldots, X_{n+\ell}) \in A \,|\, (X_0, \ldots, X_n) = (x_0, \ldots, x_n))\\ = \mathbf{P}((X_{n+1}, \ldots, X_{n+\ell}) \in A \,|\, X_n = x_n).\end{aligned}$$

Then, show that

$$\mathbf{P}((X_{n+1}, \ldots, X_{n+\ell}) \in A \,|\, X_n = x_n) = \mathbf{P}((X_1, \ldots, X_\ell) \in A \,|\, X_0 = x_n).$$

(Hint: it may be helpful to use the Multiplication Rule .)

Exercise 8. Suppose we have a Markov chain $X_0, X_1, \ldots$ with finite state space Ω. Let $y \in \Omega$. Define $L_y := \max\{n \geq 0\colon X_n = y\}$. Is L_y a stopping time? Prove your assertion.

Exercise 9. Let x, y be points in the state space of a finite Markov Chain $(X_0, X_1, \ldots)$. Let $T_y = \min\{n \geq 1 \colon X_n = y\}$ be the first arrival time of y. Let j, k be positive integers. Show that

$$\mathbf{P}_x(T_y > kj \,|\, T_y > (k-1)j) \leq \max_{z \in \Omega} \mathbf{P}_z(T_y > j).$$

(Hint: use Exercise 7)

Exercise 10. Let x, y be points in the state space of a finite Markov Chain $(X_0, X_1, \ldots)$ with transition matrix P. Let $T_y = \min\{n \geq 1 \colon X_n = y\}$ be the first arrival time of y. Let j be a positive integer. Show that

$$P^j(x, y) \leq \mathbf{P}_x(T_y \leq j).$$

(Hint: can you induct on j?)

Exercise 11. Let x, y be any states in a finite irreducible Markov chain. Show that $\mathbf{E}_x T_y < \infty$. In particular, $\mathbf{P}_y(T_y < \infty) = 1$, so all states are recurrent.

Exercise 12 (Simplified Monopoly)**.** Let $\Omega = \{1, 2, \ldots, 10\}$. We consider Ω to be the ten spaces of a circular game board. You move from one space to the next by rolling a fair six-sided die. So, for example $P(1, k) = 1/6$ for every $2 \leq k \leq 7$. More generally, for every $j \in \Omega$ with $j \neq 5$, $P(j, k) = 1/6$ if $k = (j + i) \bmod 10$ for some $1 \leq i \leq 6$. Finally, the space 5 forces you to return to 1, so that $P(5, 1) = 1$. (Note that $\bmod 10$ denotes arithmetic modulo 10, so e.g. $7 + 5 = 2 \bmod 10$.)

Using a computer, find the unique stationary distribution of this Markov chain. Which point has the highest stationary probability? The lowest?

Compare this stationary distribution to the stationary distribution that arises from the following doubly stochastic matrix: for all $j \in \Omega$, $P(j, k) = 1/6$ if $k = (j + i) \bmod 10$ for some $1 \leq i \leq 6$.