Please provide complete and well-written solutions to the following exercises.

Due April 9, 12PM noon PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

Homework 6

Exercise 1. Let $\lambda > 0$. Let $\tau_1, \tau_2, \ldots$ be independent exponential random variables with parameter λ. For any $n \geq 1$, let $T_n = \tau_1 + \cdots + \tau_n$. Fix positive integers $n_k > \cdots > n_1$ and positive real numbers $t_k > \cdots > t_1$. Then

$$f_{T_{n_k},\ldots,T_{n_1}}(t_k, \ldots, t_1) = f_{T_{(n_k - n_{k-1})}}(t_k - t_{k-1}) \cdots f_{T_{(n_2 - n_1)}}(t_2 - t_1) f_{T_{n_1}}(t_1).$$

(Hint: just try to case $k = 2$ first, and use a conditional density function.)

Exercise 2. Let $s, t > 0$ and let m, n be nonnegative integers. Let $0 < t_m < t_{m+1} < t_{m+n} < t_{m+n+1}$, and define (using the notation of Exercise 1),

$$g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) := f_{T_1}(t_{m+n+1} - t_{m+n}) f_{T_{n-1}}(t_{m+n} - t_{m+1}) f_{T_1}(t_{m+1} - t_m) f_{T_m}(t_m).$$

Let $\{N(s)\}_{s \geq 0}$ be a Poisson Process with parameter $\lambda > 0$. Show that

$$\begin{aligned}&\mathbf{P}(N(s+t) = m+n,\, N(s) = m)\\ &= \int_0^s \left(\int_s^{s+t} \left(\int_{t_{m+1}}^{s+t} \left(\int_{s+t}^{\infty} g(t_m, t_{m+1}, t_{m+n}, t_{m+n+1}) dt_{m+n+1} \right) dt_{m+n} \right) dt_{m+1} \right) dt_m.\end{aligned}$$

(Hint: use the joint density, and then use Exercise 1.)

Exercise 3. Suppose you are running a (busy) car wash. The number of red cars that come to the car wash between time 0 and time $s > 0$ is a Poisson poisson with rate 2. The number of blue cars that come to car wash between time 0 and time $s > 0$ is a Poisson poisson with rate 3. Both Poisson processes are independent of each other. All cars are either red or blue. With what probability will five blue cars arrive, before three red cars have arrived?

Exercise 4. Let m be a positive integer and let P be an $m \times m$ real matrix.

- Show that the sum
$$\sum_{k=0}^{\infty} \frac{P^k}{k!}$$
converges. That is, e^P is well-defined.
- Show that
$$e^{P+I} = e^P e^I.$$
- Find $m \times m$ matrices P, Q such that $e^{P+Q} \neq e^P e^Q$.

Exercise 5. Let m be a positive integer and let P be an $m \times m$ real matrix. Denote $H_t := e^{t(P-I)}$ for all $t \geq 0$. Let $f \in \mathbf{R}^m$ be a column vector. Then $H_t f$ denotes multiplying the matrix H_t against the vector f. Show the following:

- $H_0 = I$.
- $H_{s+t} = H_s H_t$ for all $s, t \geq 0$. (This identity is an analogue of the Chapman-Kolmogorov equation.)
- $H_t 1 = 1$ for all $t \geq 0$. (Moreover, H_t is a stochastic matrix, for all $t \geq 0$. Here 1 denotes the vector of all ones.)
- $\frac{d}{dt}|_{t=0} H_t = \lim_{t\to 0^+} \frac{H_t - H_0}{t} = (P - I)$.
- For any $f \in \mathbf{R}^m$, we have

$$\frac{d}{dt} H_t f = (P - I) H_t f, \qquad \forall\, t \geq 0.$$

Exercise 6 (Markov Property, Continuous-Time)**.** Show that a (finite) continuous-time Markov chain satisfies the following Markov property: for all $x, y \in \Omega$, for any $n \geq 1$, $t > 0$ and for any $s > s_{n-1} > \cdots > s_0 > 0$ and for all events H_{n-1} of the form $H_{n-1} = \cap_{k=0}^{n-1}\{X_{s_k} = x_k\}$, where $x_k \in \Omega$ for all $0 \leq k \leq n-1$, such that $\mathbf{P}(H_{n-1} \cap \{X_s = x\}) > 0$, we have

$$\mathbf{P}(X_{t+s} = y \,|\, H_{n-1} \cap \{X_s = x\}) = \mathbf{P}(X_t = y \,|\, X_0 = x).$$

Exercise 7. Prove the following discrete-time version the above spectral gap inequality from class.

Let P be the transition matrix of a finite, irreducible, reversible Markov chain, with state space Ω and with (unique) stationary distribution π. Let

$$\gamma_* := 1 - \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P \text{ with } \lambda \neq 1\}$$

be the absolute spectral gap of P. Then, for any $f \in \mathbf{R}^\Omega$ and for any integer $k \geq 1$,

$$\mathrm{Var}_\pi(P^k f) \leq (1 - \gamma_*)^{2k} \mathrm{Var}_\pi f.$$

Exercise 8 (Scaling Invariance)**.** Let $a > 0$. Let $\{B(t)\}_{t \geq 0}$ be a standard Brownian motion. For any $t > 0$, define $X(t) := \frac{1}{\sqrt{a}} B(at)$. Show that $\{X(t)\}_{t\geq 0}$ is also a standard Brownian motion.

Exercise 9. Let $x_1, \ldots, x_n \in \mathbf{R}$, and if $t_n > \cdots > t_1 > 0$. Using the independent increment property, show that the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

has a multivariate normal distribution. That is, the joint density of $(B(t_1), \ldots, B(t_n))$ is

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}, \qquad \forall x \in \mathbf{R},\ t > 0.$$