Graduate Probability, Fall 2020, USC — Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 1

This exam contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 60 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
Total:	60	

1. (15 points) Let $X_1, X_2, \ldots \Omega \to \mathbf{R}$ be random variables such that $\mathbf{E}X_i = 0$ and $\mathbf{E}X_i^2 = 1$ for all $i \geq 1$. Show that

$$\mathbf{P}(X_n > n \text{ for infinitely many } n \geq 1) = 0.$$

2. (15 points) In this Exercise we will use the following form of Jensen's inequality (which you can take as a given fact):

   Let $X\colon \Omega \to [-\infty, \infty]$ be a random variable. Let $\phi\colon \mathbf{R} \to \mathbf{R}$ be convex. Assume that $\mathbf{E}\,|X| < \infty$ and $\mathbf{E}\,|\phi(X)| < \infty$. Then

$$\phi(\mathbf{E}X) \leq \mathbf{E}\phi(X).$$

   In this Exercise, the above form of Jensen's inequality is the **only** form of Jensen's inequality that you are allowed to use.

   Prove: if $\mathbf{E}(X^2) < \infty$, then $|\mathbf{E}X| < \infty$.

3. (15 points) Let Ω be a universal set, and let $\mathcal{F}$ be a set of subsets of Ω. Suppose $\mathcal{F}$ is a monotone class.

   (As usual, we define $\sigma(\mathcal{F})$ to be the σ-algebra generated by $\mathcal{F}$. That is, $\sigma(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$.)

   Prove or disprove the following statement:

   $\sigma(\mathcal{F})$ is the smallest monotone class containing $\mathcal{F}$.

4. (15 points) Give an example of a function $f\colon [-1,1]\times[-1,1]\to \mathbf{R}$ such that

- For a.e. $x\in[-1,1]$, $\int_{-1}^{1} f(x,y)dy = 0$.
- For a.e. $y\in[-1,1]$, $\int_{-1}^{1} f(x,y)dx = 0$.
- $\int_{[-1,1]\times[-1,1]} |f(x,y)|\, dxdy = \infty$.

That is, $\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)dy\right)dx = 0$, $\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)dx\right)dy = 0$, while

$$\int_{[-1,1]\times[-1,1]} |f(x,y)|\, dxdy = \infty.$$

Consequently, a "converse" of Fubini's Theorem does not hold for probability spaces. (We can divide by appropriate constants to turn these integrals into expected values.)

(Scratch paper)