Graduate Probability, Fall 2020, USC Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books and notes on this exam. You cannot use a calculator or any other electronic device (or internet-enabled device) on this exam. You are required to show your work on each problem on the exam. The following rules apply:

- You have 60 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper is at the end of the exam.

Do not write in the table to the right. Good luck![a]

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be a sequence of independent, identically distributed random variables such that

$$\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2, \qquad \forall\, n \geq 1.$$

Compute

$$\mathbf{P}\Big(\limsup_{n\to\infty} X_n > 0\Big).$$

2. (10 points) Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be a sequence of mean zero independent random variables such that: $\mathbf{E}X_n^2 \leq 10$ for all $n \geq 1$. For any $m \geq 1$, define

$$Y_m := \sum_{n=1}^{m} \frac{X_n}{n^{2/3}}.$$

Show that $Y_1, Y_2, \ldots$ converges almost surely.

3. (10 points) Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be a sequence of independent, identically distributed random variables such that

$$\mathbf{P}(X_n \leq t) = \int_{-\infty}^{t} \frac{1}{\pi} \frac{1}{1+x^2} dx, \qquad \forall\, t \in \mathbf{R},\, \forall\, n \geq 1.$$

- Does there exist a sequence of positive real numbers $b_1, b_2, \ldots$ such that

$$\frac{1}{b_n}(X_1 + \cdots + X_n)$$

  converges in probability to 1 as $n \to \infty$? Justify your answer. [Hint: You may freely use that $\phi_{X_1}(t) = \mathbf{E}e^{\sqrt{-1}X_1 t} = e^{-|t|}$ for all $t \in \mathbf{R}$. What is the Fourier transform of $X_1 + \cdots + X_n$? What does this imply about the distribution of $X_1 + \cdots + X_n$?]
- Does there exist a sequence of real numbers $c_1, c_2, \ldots$ such that $\frac{1}{n}(X_1 + \cdots + X_n) - c_n$ converges in probability to 0 as $n \to \infty$? Justify your answer.

4. (10 points) Prove or disprove the following statement:

   Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be random variables. Assume that the laws $\mu_{X_1}, \mu_{X_2}, \ldots$ converge vaguely to a measure μ on Ω. Then μ is a probability measure.

5. (10 points) Let $n \geq 1$. Let $X_1, \ldots, X_n$ be independent random variables such that $\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2$, $\forall\, n \geq 1$. Let $Y_1, \ldots, Y_n$ be independent standard Gaussian random variables that are also independent of $X_1, \ldots, X_n$. Note that $\mathbf{E}X_1 = \mathbf{E}Y_1$ and $\mathbf{E}X_1^2 = \mathbf{E}Y_1^2$. Denote $X := (X_1, \ldots, X_n)$, $Y := (Y_1, \ldots, Y_n)$. For any $1 \leq j \leq n$, let $W_j := (X_1, \ldots, X_{j-1}, Y_j, \ldots, Y_n)$.

   Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a multilinear polynomial. That is, for any subset $S \subseteq \{1, \ldots, n\}$, there exist (deterministic) constants c_S such that

$$f(z) = \sum_{S \subseteq \{1,\ldots,n\}} c_S \prod_{i \in S} z_i, \qquad \forall\, z = (z_1, \ldots, z_n) \in \mathbf{R}^n.$$

   (We define $\prod_{i \in \emptyset} z_i := 1$.) Finally, let $\phi\colon \mathbf{R} \to \mathbf{R}$ be a Schwartz function. Prove:

$$|\mathbf{E}\phi(f(X)) - \mathbf{E}\phi(f(Y))| \leq \sup_{t \in \mathbf{R}} |\phi'''(t)| \cdot 100 \sum_{j=1}^{n} \mathbf{E}\Big(\frac{\partial}{\partial z_j} f(W_j)\Big)^3$$

   (Hint: argue as in Lindeberg's proof of the Central Limit Theorem.)

   (Hint: Let $f_j(z) := \sum_{S \subseteq \{1,\ldots,n\}\colon j \in S} c_S \prod_{i \in S \setminus \{j\}} z_i$, $g_j(z) := \sum_{S \subseteq \{1,\ldots,n\}\colon j \notin S} c_S \prod_{i \in S} z_i$, for any $1 \leq j \leq n$. Note that $\frac{\partial}{\partial z_j} f(z) = f_j(z)$ and $f(z) = z_j f_j(z) + g_j(z)$. Note also that $f_j(z)$ has no z_j terms inside it.)

(Scratch paper)