507A Midterm 2 Solutions[1]

1. Question 1

Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be a sequence of independent, identically distributed random variables such that

$$\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2, \qquad \forall\, n \geq 1.$$

Compute

$$\mathbf{P}\Big(\limsup_{n\to\infty} X_n > 0\Big).$$

Solution. Note that $\sum_{n=1}^{\infty} \mathbf{P}(X_n = 1) = \infty$. So, from the Second Borel-Cantelli Lemma, the probability that $X_n = 1$ for infinitely many $n \geq 1$ is 1. Therefore, $\mathbf{P}(\limsup_{n\to\infty} X_n = 1) = 1$.

2. Question 2

Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be a sequence of mean zero independent random variables such that: $\mathbf{E}X_n^2 \leq 10$ for all $n \geq 1$. For any $m \geq 1$, define

$$Y_m := \sum_{n=1}^{m} \frac{X_n}{n^{2/3}}.$$

Show that $Y_1, Y_2, \ldots$ converges almost surely.

Solution. By assumption, $\mathrm{var}(X_n) \leq \mathbf{E}X_n^2 \leq 10$ for all $n \geq 1$. Note also that $\sum_{n=1}^{\infty} \mathrm{var}(X_n/n^{2/3}) \leq 10 \sum_{n=1}^{\infty} n^{-4/3} < \infty$. So, we are done by Theorem 2.25 in the notes.

3. Question 3

Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be a sequence of independent, identically distributed random variables such that

$$\mathbf{P}(X_n \leq t) = \int_{-\infty}^{t} \frac{1}{\pi} \frac{1}{1+x^2} dx, \qquad \forall\, t \in \mathbf{R},\, \forall\, n \geq 1.$$

- Does there exist a sequence of positive real numbers $b_1, b_2, \ldots$ such that

  $$\frac{1}{b_n}(X_1 + \cdots + X_n)$$

  converges in probability to 1 as $n \to \infty$? Justify your answer. [Hint: You may freely use that $\phi_{X_1}(t) = \mathbf{E}e^{\sqrt{-1}X_1 t} = e^{-|t|}$ for all $t \in \mathbf{R}$. What is the Fourier transform of $X_1 + \cdots + X_n$? What does this imply about the distribution of $X_1 + \cdots + X_n$?]
- Does there exist a sequence of real numbers $c_1, c_2, \ldots$ such that $\frac{1}{n}(X_1 + \cdots + X_n) - c_n$ converges in probability to 0 as $n \to \infty$? Justify your answer.

Solution. Since $X_1, \ldots, X_n$ are independent,

$$\phi_{(X_1+\cdots+X_n)/b_n}(t) = \prod_{j=1}^{n} \phi_{X_j/b_n}(t) = [\phi_{X_1}(t/b_n)]^n = e^{-|t|n/b_n} = \phi_{X_1}(tn/b_n) = \phi_{X_1 n/b_n}.$$

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By Remark 3.21 in the notes, we conclude that $(X_1 + \cdots + X_n)/b_n$ is equal in distribution to $X_1 n/b_n$. So, it is impossible for $(X_1 + \cdots + X_n)/b_n$ to converge in probability to the constant random variable 1. By similar reasoning, $\frac{1}{n}(X_1 + \cdots + X_n)$ is equal in distribution to X_1, and it is impossible for $X_1 - c_n$ to converge in probability to 0, so the same is true for $\frac{1}{n}(X_1 + \cdots + X_n)$.

4. Question 4

Prove or disprove the following statement:

Let $X_1, X_2, \ldots : \Omega \to \mathbf{R}$ be random variables. Assume that the laws $\mu_{X_1}, \mu_{X_2}, \ldots$ converge vaguely to a measure μ on Ω. Then μ is a probability measure.

Solution. This statement is false, as we showed in the notes on page 37.

5. Question 5

Let $n \geq 1$. Let $X_1, \ldots, X_n$ be independent random variables such that $\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = 1/2$, $\forall\, n \geq 1$. Let $Y_1, \ldots, Y_n$ be independent standard Gaussian random variables that are also independent of $X_1, \ldots, X_n$. Note that $\mathbf{E}X_1 = \mathbf{E}Y_1$ and $\mathbf{E}X_1^2 = \mathbf{E}Y_1^2$. Denote $X := (X_1, \ldots, X_n)$, $Y := (Y_1, \ldots, Y_n)$. For any $1 \leq j \leq n$, let $W_j := (X_1, \ldots, X_{j-1}, Y_j, \ldots, Y_n)$.

Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a multilinear polynomial. That is, for any subset $S \subseteq \{1, \ldots, n\}$, there exist (deterministic) constants c_S such that

$$f(z) = \sum_{S \subseteq \{1,\ldots,n\}} c_S \prod_{i \in S} z_i, \qquad \forall\, z = (z_1, \ldots, z_n) \in \mathbf{R}^n.$$

(We define $\prod_{i \in \emptyset} z_i := 1$.) Finally, let $\phi\colon \mathbf{R} \to \mathbf{R}$ be a Schwartz function. Prove:

$$|\mathbf{E}\phi(f(X)) - \mathbf{E}\phi(f(Y))| \leq \sup_{t \in \mathbf{R}} |\phi'''(t)| \cdot 100 \sum_{j=1}^{n} \mathbf{E}\Big(\frac{\partial}{\partial z_j} f(W_j)\Big)^3$$

(Hint: argue as in Lindeberg's proof of the Central Limit Theorem.)

(Hint: Let $f_j(z) := \sum_{S \subseteq \{1,\ldots,n\}\colon j \in S} c_S \prod_{i \in S \setminus \{j\}} z_i$, $g_j(z) := \sum_{S \subseteq \{1,\ldots,n\}\colon j \notin S} c_S \prod_{i \in S} z_i$, for any $1 \leq j \leq n$. Note that $\frac{\partial}{\partial z_j} f(z) = f_j(z)$ and $f(z) = z_j f_j(z) + g_j(z)$. Note also that $f_j(z)$ has no z_j terms inside it.) *Solution.* We write a telescoping sum

$$\begin{aligned}
&|\mathbf{E}\phi(f(X)) - \mathbf{E}\phi(f(Y))| \\
&\quad = \left|\sum_{j=1}^{n} [\mathbf{E}\phi(f(X_1, \ldots, X_{j-1}, Y_j, \ldots, Y_n)) - \mathbf{E}\phi(f(X_1, \ldots, X_j, Y_{j+1}, \ldots, Y_n))]\right| \\
&\quad \leq \sum_{j=1}^{n} |\mathbf{E}\phi(f(X_1, \ldots, X_{j-1}, Y_j, \ldots, Y_n)) - \mathbf{E}\phi(f(X_1, \ldots, X_j, Y_{j+1}, \ldots, Y_n))|\,.
\end{aligned}$$

We then need to bound $|\mathbf{E}\phi(f(W_{j-1})) - \mathbf{E}\phi(f(W_j))|$. Using the definition of g_j, f_j

$$\mathbf{E}\phi(f(W_{j-1})) - \mathbf{E}\phi(f(W_j)) = \mathbf{E}\phi(g_j(W_{j-1}) + Y_j f_j(W_{j-1})) - \mathbf{E}\phi(g_j(W_{j-1}) + X_j f_j(W_{j-1}))$$

Using a third order Taylor expansion for ϕ, in the form $\phi(t+y) = \phi(t)+y\phi'(t)+(y^2/2)\phi''(t)+(y^3/6)\phi'''(c)$ for some c between t and $t+y$, we have

$$\begin{aligned}
&\mathbf{E}\phi(g_j(W_{j-1}) + Y_j f_j(W_{j-1})) - \mathbf{E}\phi(g_j(W_{j-1}) + X_j f_j(W_{j-1})) \\
&= \mathbf{E}\phi(g_j(W_{j-1})) + \mathbf{E}Y_j f_j(W_{j-1})) + \mathbf{E}Y_j^2[f_j(W_{j-1})]^2 + (1/6)\mathbf{E}Y_j^3[f_j(W_{j-1})]^3\mathbf{E}\phi'''(B_j) \\
&\qquad - \Big[\mathbf{E}\phi(g_j(W_{j-1})) + \mathbf{E}X_j f_j(W_{j-1})) + \mathbf{E}X_j^2[f_j(W_{j-1})]^2 + (1/6)\mathbf{E}X_j^3[f_j(W_{j-1})]^3\mathbf{E}\phi'''(C_j)\Big]
\end{aligned}$$

Since f_j contains no X_j terms, independent implies that $\mathbf{E}Y_j f_j(W_{j-1})) = \mathbf{E}Y_j\mathbf{E}f_j(W_{j-1})) = 0$. Similarly, $\mathbf{E}X_j f_j(W_{j-1})) = \mathbf{E}X_j\mathbf{E}f_j(W_{j-1})) = 0$. Similarly, by independence, $\mathbf{E}Y_j^2[f_j(W_{j-1})]^2 = \mathbf{E}Y_j^2\mathbf{E}[f_j(W_{j-1})]^2 = \mathbf{E}[f_j(W_{j-1})]^2$ and $\mathbf{E}X_j^2[f_j(W_{j-1})]^2 = \mathbf{E}X_j^2\mathbf{E}[f_j(W_{j-1})]^2 = \mathbf{E}[f_j(W_{j-1})]^2$. So, all the lower order terms cancel and we get

$$\begin{aligned}
|\mathbf{E}\phi(f(W_{j-1})) - \mathbf{E}\phi(f(W_j))| &\leq \sup_{t\in\mathbf{R}} |\phi'''(t)|\,(1/6)\mathbf{E}\,|Y_j|^3\,\mathbf{E}\,|f_j(W_{j-1})|^3 \\
&\quad + \sup_{t\in\mathbf{R}} |\phi'''(t)|\,(1/6)\mathbf{E}\,|X_j|^3\,\mathbf{E}\,|f_j(W_{j-1})|^3 .
\end{aligned}$$

So, we conclude since $\mathbf{E}\,|X_j|^3 = 1$ and $\mathbf{E}\,|Y_j|^3 \leq 10$.