

Please provide complete and well-written solutions to the following exercises.

Due September 4, 12PM noon PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

Homework 2

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a finite or countable probability space. If $X: \Omega \rightarrow [0, \infty]$ is a random variable with $\mathbf{E}|X| < \infty$, show that

$$\mathbf{E}X = \sum_{\omega \in \Omega} X(\omega) \mathbf{P}(\omega).$$

Exercise 2. Let X, Y be random variables such that $X, Y \geq 0$ or $\mathbf{E}|X|, \mathbf{E}|Y| < \infty$. For any $a \in [-\infty, \infty]$, define $0 \cdot a := 0$. Show:

- $\mathbf{E}(X + Y) = \mathbf{E}X + \mathbf{E}Y$ and if $c \in \mathbf{R}$, then $\mathbf{E}(cX) = c\mathbf{E}X$.
- If $\mathbf{P}(X = Y) = 1$, then $\mathbf{E}X = \mathbf{E}Y$.
- $\mathbf{E}|X| \geq 0$ with equality only when $X = 0$ almost surely.
- If $X \leq Y$ almost surely, then $\mathbf{E}X \leq \mathbf{E}Y$.

Exercise 3 (Inclusion-Exclusion Formula). Let $A_1, \dots, A_n \subseteq \Omega$ be events. Then:

$$\begin{aligned} \mathbf{P}(\cup_{i=1}^n A_i) &= \sum_{i=1}^n \mathbf{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathbf{P}(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} \mathbf{P}(A_i \cap A_j \cap A_k) \\ &\quad \dots + (-1)^{n+1} \mathbf{P}(A_1 \cap \dots \cap A_n). \end{aligned}$$

To prove this formula, show that $1_{\cup_{i=1}^n A_i} = 1 - \prod_{i=1}^n (1 - 1_{A_i})$ and then take expected values of both sides.

Exercise 4. Let $\phi: \mathbf{R} \rightarrow \mathbf{R}$. We say that ϕ is **convex** if, for any $x, y \in \mathbf{R}$ and for any $t \in [0, 1]$, we have

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y).$$

Let $\phi: \mathbf{R} \rightarrow \mathbf{R}$. Show that ϕ is convex if and only if: for any $y \in \mathbf{R}$, there exists a constant a and there exists a function $L: \mathbf{R} \rightarrow \mathbf{R}$ defined by $L(x) = a(x - y) + \phi(y)$, $x \in \mathbf{R}$, such that $L(y) = \phi(y)$ and such that $L(x) \leq \phi(x)$ for all $x \in \mathbf{R}$. (In the case that ϕ is differentiable, the latter condition says that ϕ lies above all of its tangent lines.)

(Hint: Suppose ϕ is convex. If x is fixed and y varies, show that $\frac{\phi(y) - \phi(x)}{y - x}$ increases as y increases. Draw a picture. What slope a should L have at x ?)

Exercise 5 (Jensen's Inequality). Let $X: \Omega \rightarrow [-\infty, \infty]$ be a random variable. Let $\phi: \mathbf{R} \rightarrow \mathbf{R}$ be convex. Assume that $\mathbf{E}|X| < \infty$ and $\mathbf{E}|\phi(X)| < \infty$. Then

$$\phi(\mathbf{E}X) \leq \mathbf{E}\phi(X).$$

(Hint: use Exercise 4 with $y := \mathbf{E}X$.) Deduce the **triangle inequality**:

$$|\mathbf{E}X| \leq \mathbf{E}|X|.$$

Exercise 6 (Markov's Inequality). Let $X: \Omega \rightarrow [-\infty, \infty]$ be a random variable. Then

$$\mathbf{P}(|X| \geq t) \leq \frac{\mathbf{E}|X|}{t}, \quad \forall t > 0.$$

(Hint: multiply both sides by t and use monotonicity of \mathbf{E} .)

Exercise 7. Combining Jensen's Inequality with the Monotone Convergence Theorem, show that if $\mathbf{E}X^2 < \infty$, then $\mathbf{E}|X| < \infty$, so $\mathbf{E}X \in \mathbf{R}$.

Exercise 8. Let $a, b \in \mathbf{R}$ and let $X: \Omega \rightarrow [-\infty, \infty]$ be a random variable with $\mathbf{E}X^2 < \infty$. Show that

$$\text{var}(aX + b) = a^2 \text{var}(X).$$

Then, let X be a standard Gaussian. Show that $\mathbf{E}X = 0$ and $\text{var}(X) = 1$.

Finally, show that the quantity $\mathbf{E}(X - t)^2$ is minimized for $t \in \mathbf{R}$ uniquely when $t = \mathbf{E}X$.

Exercise 9 (The Chernoff Bound). Let $X: \Omega \rightarrow [-\infty, \infty]$ be a random variable. Show that, for any $r, t > 0$,

$$\mathbf{P}(X > t) \leq e^{-rt} \mathbf{E}e^{rX}.$$

Exercise 10 (Paley-Zygmund Inequality). Let X be a nonnegative random variable with $\mathbf{E}X^2 < \infty$. Let $0 \leq t \leq 1$. Then

$$\mathbf{P}(X > t \mathbf{E}X) \geq (1 - t)^2 \frac{(\mathbf{E}X)^2}{\mathbf{E}X^2}.$$

(Hint: Apply the Cauchy-Schwarz inequality to $X1_{\{X > t\mathbf{E}X\}}$.)