Please provide complete and well-written solutions to the following exercises.

Due September 11, 12PM noon PST, to be uploaded as a single PDF document to blackboard (under the Assignments tab).

Homework 3

Exercise 1 (Stein Identity)**.** Let X be a standard Gaussian random variable, so that X has density $x \mapsto e^{-x^2/2}/\sqrt{2\pi}$, $\forall\, x \in \mathbf{R}$. Let $g\colon \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function such that g and g' have polynomial volume growth. That is, $\exists\, a, b > 0$ such that $|g(x)|, |g'(x)| \leq a(1 + |x|)^b$, $\forall\, x \in \mathbf{R}$. Prove the **Stein identity**

$$\mathbf{E}Xg(X) = \mathbf{E}g'(X).$$

Using this identity, recursively compute $\mathbf{E}X^k$ for any positive integer k.

Alternatively, for any $t > 0$, show that $\mathbf{E}e^{tX} = e^{t^2/2}$, i.e. compute the **moment generating function** of X. Then, using $\frac{d^k}{dt^k}|_{t=0}\mathbf{E}e^{tX} = \mathbf{E}X^k$ and using the power series expansion of the exponential, compute $\mathbf{E}X^k$ directly from the identity $\mathbf{E}e^{tX} = e^{t^2/2}$.

Exercise 2 (Finite Product Measure)**.** Let $(\Omega_i, \mathcal{F}_i, \mu_i)$ be probability spaces for any $1 \leq i \leq n$. Show that there exists a unique probability measure, denoted $\prod_{i=1}^n \mu_i$ on $(\prod_{i=1}^n \Omega_i, \prod_{i=1}^n \mathcal{F}_i)$ (where the latter measurable space is defined in the notes) such that

$$\Big(\prod_{i=1}^n \mu_i\Big)\Big(\prod_{i=1}^n A_i\Big) = \prod_{i=1}^n \mu_i(A_i), \qquad \forall\, A_1 \in \mathcal{F}_1, \ldots, A_n \in \mathcal{F}_n.$$

Exercise 3. Let $X\colon \Omega \to \mathbf{R}$ be a random variable (as usual $\mathbf{R}$ has the Borel σ-algebra). Show that X is independent of itself if and only if X is almost surely constant.

Also, show that a constant random variable is independent of any other random variable.

Exercise 4. Let $X_1, \ldots, X_n$ be discrete random variables (i.e. they take values in finite or countable spaces $S_1, \ldots, S_n$ with their discrete σ-algebras). Show that $X_1, \ldots, X_n$ are independent if and only if:

$$\mathbf{P}\Big(\bigcap_{i=1}^n \{X_i = x_i\}\Big) = \prod_{i=1}^n \mathbf{P}(X_i = x_i), \qquad \forall\, x_1 \in S_1, \ldots, x_n \in S_n.$$

Exercise 5. Show that $X_1, \ldots, X_n\colon \Omega \to \mathbf{R}$ are independent if and only if:

$$\mathbf{P}\Big(\bigcap_{i=1}^n \{X_i \leq x_i\}\Big) = \prod_{i=1}^n \mathbf{P}(X_i \leq x_i), \qquad \forall\, x_1, \ldots, x_n \in \mathbf{R}.$$

Exercise 6 (Optional)**.** Let V be a finite-dimensional vector space over a finite field $\mathbb{F}$. Let X be a random variable uniformly distributed in V. Let $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ be a non-degenerate bilinear form on V (if $v \in V$ satisfies $\langle v, w \rangle = 0$ for all $w \in V$, then $v = 0$.). Let

$v_1, \ldots, v_n$ be non-zero vectors in V. Show that the random variables $\langle X, v_1\rangle, \ldots, \langle X, v_n\rangle$ are independent if and only if the vectors $v_1, \ldots, v_n$ are linearly independent.

Exercise 7. Give an example of three random variables $X, Y, Z \colon \Omega \to [-\infty, \infty]$ that are pairwise independent (any two of the random variables X, Y, Z are independent of each other), but such that X, Y, Z are not independent. (Hint: Exercise 6 might be helpful.)

Exercise 8 (Optional)**.** Let $X \colon \Omega \to \mathbf{R}^n$ be a random variable with the **standard Gaussian distribution**:

$$\mathbf{P}(X \in A) := \int_A e^{-(x_1^2+\cdots+x_n^2)/2} dx (2\pi)^{-n/2}, \qquad \forall\, A \subseteq \mathbf{R}^n \text{ measurable.}$$

Let $v_1, \ldots, v_m$ be vectors in $\mathbf{R}^n$. Let $\langle \cdot, \cdot \rangle : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be the standard inner product on $\mathbf{R}^n$, so that $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$ for any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. Show that the random variables $\langle X, v_1\rangle, \ldots, \langle X, v_m\rangle$ are independent if and only if the vectors $v_1, \ldots, v_m$ are pairwise orthogonal.

Exercise 9.

- Show that two events A, B are independent if and only if $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$
- Find events A, B, C such that $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$, but such that A, B, C are not independent.
- Find events A, B, C that are pairwise independent (so that any two of A, B, C are independent), but such that A, B, C are not independent.