

## 541A Midterm 1 Solutions<sup>1</sup>

### 1. QUESTION 1

Two people take turns throwing darts at a board. Person  $A$  goes first, and each of her throws has a probability of  $1/4$  of hitting the bullseye. Person  $B$  goes next, and each of her throws has a probability of  $1/3$  of hitting the bullseye. Then Person  $A$  goes, and so on. With what probability will Person  $A$  hit the bullseye before Person  $B$  does?

*Solution.* Person  $A$  hits the bullseye on her first try with probability  $1/4$ . If both  $A$  and  $B$  miss their first throw, then Person  $A$  hits the bullseye on her second try with probability  $(1 - 1/4)(1 - 1/3)(1/4)$ . If both  $A$  and  $B$  miss their first two throws, then Person  $A$  hits the bullseye on her third try with probability  $(1 - 1/4)^2(1 - 1/3)^2(1/4)$ . For any positive integer  $k$ , let  $C_k$  be the event that both  $A$  and  $B$  miss their first  $k$  throws, and Person  $A$  hits the bullseye on the  $(k + 1)^{st}$  try. Then  $\mathbf{P}(C_k) = (1 - 1/4)^k(1 - 1/3)^k(1/4) = (3/4)^k(2/3)^k(1/4) = (1/2)^k(1/4)$ . Let  $C$  be the event that person  $A$  hits the bullseye before person  $B$ . Then  $C = \cup_{k \geq 0} C_k$ , and  $C_k \cap C_{k'} = \emptyset$  if  $k \neq k'$ . So, from the axioms for a probability law,

$$\mathbf{P}(C) = \mathbf{P}(\cup_{k \geq 0} C_k) = \sum_{k=0}^{\infty} \mathbf{P}(C_k) = (1/4) \sum_{k=0}^{\infty} (1/2)^k = (1/4)(2) = 1/2.$$

### 2. QUESTION 2

Let  $X: \Omega \rightarrow \mathbf{R}$  be a random variable. Prove:

$$\mathbf{E}(e^X) \geq e^{\mathbf{E}X}.$$

*Solution.* Let  $y = \mathbf{E}X$ . For any  $x \in \mathbf{R}$ , let  $f(x) = e^x$ , so that  $f: \mathbf{R} \rightarrow \mathbf{R}$ . Note that  $f$  is convex, since  $f''(x) = e^x > 0$  for any  $x \in \mathbf{R}$ . Since  $f$  is convex, the function  $f$  lies above any of its tangent lines. That is,  $f(x) \geq f(y) + f'(y)(x - y)$ , for all  $x \in \mathbf{R}$ . Since we chose  $y = \mathbf{E}X$ , we have  $f(x) \geq f(\mathbf{E}X) + f'(\mathbf{E}X)(x - \mathbf{E}X)$ , for all  $x \in \mathbf{R}$ . Taking expectation with respect to  $x = X$ , we have  $\mathbf{E}f(X) \geq f(\mathbf{E}X) + f'(\mathbf{E}X)(\mathbf{E}X - \mathbf{E}X) = f(\mathbf{E}X)$ . That is,  $\mathbf{E}e^X \geq e^{\mathbf{E}X}$ , as desired.

### 3. QUESTION 3

Suppose you flip a fair coin 80 times. During each coin flip, this coin has probability  $1/2$  of landing heads, and probability  $1/2$  of landing tails.

Let  $A$  be the event that you get more than 50 heads in total. Show that

$$\mathbf{P}(A) \leq \frac{1}{10}.$$

*Solution 1.* For any  $n \geq 1$ , define  $X_n$  so that

$$X_n = \begin{cases} 1 & , \text{ if the } n^{th} \text{ coin flip is heads} \\ 0 & , \text{ if the } n^{th} \text{ coin flip is tails.} \end{cases}$$

By its definition  $\mathbf{E}X_n = 1/2$  and  $\text{var}(X_n) = (1/2)(1/4) + (1/2)(1/4) = 1/4$ .

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Let  $S := X_1 + \dots + X_{80}$  be the number of heads that are flipped. Then  $\mathbf{E}S = 40$ , and  $\text{var}(S) = 80\text{var}(X_1) = 20$ . Markov's inequality says, for any  $t > 0$

$$\mathbf{P}(S > t) \leq \mathbf{E}S/t = 40/t.$$

This is not helpful. Instead, we use Chebyshev's inequality. This says, for any  $t > 0$ ,

$$\mathbf{P}(|S - 40| > t) \leq t^{-2}\text{var}(S) = 20t^{-2}.$$

Choosing  $t = 10$  shows that  $\mathbf{P}(|S - 40| > 10) \leq 1/5$ . Now, using symmetry of  $S$  (interchanging the roles of heads and tails),

$$\mathbf{P}(|S - 40| > 10) = \mathbf{P}(S < 30) + \mathbf{P}(S > 50) = 2\mathbf{P}(S > 50).$$

So,

$$2\mathbf{P}(S > 50) = \mathbf{P}(|S - 40| > 10) \leq 1/5.$$

*Solution 2.* We use the notation of Solution 1, but instead of Chebyshev's inequality, we use the Chernoff bound. Since  $S$  is a sum of 80 independent identically distributed random variables, Proposition 2.43 from the notes says

$$M_S(t) = (M_{X_1}(t))^{80}, \quad \forall t \in \mathbf{R}.$$

So, the Chernoff bound says, for any  $r, t > 0$ ,

$$\mathbf{P}(S > r) \leq e^{-tr}(M_{X_1}(t))^{80} = e^{-tr}((1/2)(1 + e^t))^{80} \quad (*).$$

Setting  $f(t) = e^{-rt}(1 + e^t)^{80}$  and solving  $f'(t) = 0$  for  $t$  shows that  $t = \log(5/3)$  minimizes the quantity  $f(t)$ . So, choosing  $r = 50$  and  $t = \log(5/3)$  in (\*) gives

$$\mathbf{P}(S > 50) \leq e^{-tr}((1/2)(1 + 5/3))^{80} = (5/3)^{-50}(4/3)^{80} \leq 0.08 < 1/10.$$

*Solution 3.* (The following solution based on the Central Limit Theorem only received partial credit, since it only approximately shows that  $\mathbf{P}(A) < 1/10$ .) We use the notation of Solution 1, but instead of Chebyshev's inequality, we use the Central Limit Theorem. Since  $X_1, X_2, \dots$  are independent identically distributed random variables with mean  $1/2$  and variance  $1/4$ , the Central Limit Theorem implies that

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(\frac{X_1 + \dots + X_n - n/2}{\sqrt{(1/4)\sqrt{n}}} > t\right) = \int_t^\infty e^{-x^2/2} dx / \sqrt{2\pi}.$$

So, choosing  $n = 80$  and  $t = \sqrt{5}$ , we have the approximation

$$\mathbf{P}\left(\frac{X_1 + \dots + X_{80} - 40}{\sqrt{(1/4)\sqrt{80}}} > \sqrt{5}\right) \approx \int_{\sqrt{5}}^\infty e^{-x^2/2} dx / \sqrt{2\pi}.$$

Simplifying a bit,

$$\mathbf{P}(S - 40 > 10) \approx \int_{\sqrt{5}}^\infty e^{-x^2/2} dx / \sqrt{2\pi}.$$

Using  $\sqrt{5} > 2$  and the approximation  $\int_2^\infty e^{-x^2/2} dx / \sqrt{2\pi} \approx .025$ , we have

$$\mathbf{P}(S > 50) \approx \int_{\sqrt{5}}^\infty e^{-x^2/2} dx / \sqrt{2\pi} \leq \int_2^\infty e^{-x^2/2} dx / \sqrt{2\pi} \approx .025 < 1/10.$$

**Note:** it is possible to make this argument completely rigorous using the Berry-Esseen Central Limit Theorem. We have

$$\left| \mathbf{P}(S > 50) - \int_{\sqrt{5}}^{\infty} e^{-x^2/2} dx / \sqrt{2\pi} \right| \leq \frac{1}{2\sqrt{80}} \frac{\mathbf{E}|X_1 - 1/2|^3}{(\mathbf{E}|X_1 - 1/2|^2)^{3/2}} = \frac{1}{2\sqrt{80}} \approx .0559.$$

Therefore,  $\mathbf{P}(S > 50) \leq \int_{\sqrt{5}}^{\infty} e^{-x^2/2} dx / \sqrt{2\pi} + .06 \leq .03 + .06 < 1/10$ .

#### 4. QUESTION 4

Compute  $\mathbf{E}X^2$ , by differentiating the exponential family, where

$$f_w(x) := h(x) \exp\left(\sum_{i=1}^2 w_i t_i(x) - a(w)\right) \quad \forall x \in \mathbf{R}, \quad \forall w = (w_1, w_2).$$

$$a(w) = \log \int_{\mathbf{R}} h(x) \exp\left(\sum_{i=1}^2 w_i t_i(x)\right) d\mu(x), \quad \forall w = (w_1, w_2).$$

Recall that in this case,

$$t_1(x) := x, \quad t_2(x) := x^2, \quad w_1 := \frac{\mu}{\sigma^2}, \quad w_2 := -\frac{1}{2\sigma^2},$$

$$a(w) := -\frac{w_1^2}{4w_2} - \frac{1}{2} \log(-2w_2), \quad h(x) = (2\pi)^{-1/2}.$$

*Solution 1.* By differentiating  $a(w)$  twice, we get

$$\begin{aligned} e^{-a(w)} \frac{\partial^2}{\partial w_1^2} e^{a(w)} &= \frac{\partial^2}{\partial w_1^2} a(w) + \left(\frac{\partial}{\partial w_1} a(w)\right)^2 = -\frac{1}{2w_2} + \frac{w_1^2}{4w_2^2} \\ &= \int_{\mathbf{R}} x^2 h(x) \exp\left(\sum_{i=1}^2 w_i t_i(x) - a(w)\right) d\mu(x) = \mathbf{E}X^2, \end{aligned}$$

So,

$$\mathbf{E}X^2 = \sigma^2 + \mu^2 \sigma^{-4} \sigma^4 = \sigma^2 + \mu^2.$$

*Solution 2.* By differentiating  $a(w)$  once with respect to  $w_2$ , we get

$$\begin{aligned} e^{-a(w)} \frac{\partial}{\partial w_2} e^{a(w)} &= \frac{\partial}{\partial w_2} a(w) = \frac{w_1^2}{4w_2^2} - \frac{1}{w_2} \\ &= \int_{\mathbf{R}} x^2 h(x) \exp\left(\sum_{i=1}^2 w_i t_i(x) - a(w)\right) d\mu(x) = \mathbf{E}X^2, \end{aligned}$$

So,

$$\mathbf{E}X^2 = \sigma^2 + \mu^2.$$