541A Midterm 2 Solutions[1]

1. Question 1

True/False

(a) Let $n \geq 2$ be an integer. Let $X_1, \ldots, X_n$ be a random sample of size n from the Gaussian distribution with mean $\mu \in \mathbf{R}$ and variance $\sigma^2 > 0$. Let $\overline{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$. and let $S := \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$. Then $\overline{X}$ and S are independent random variables.

True; from notes, Proposition 4.7. This also follows from Basu's Theorem (as in part (e)), since $\overline{X}$ is complete and sufficient for μ, while S is ancillary for μ, so $\overline{X}$ and S are independent.

(b) A complete sufficient statistic always exists. (In your answer here, you are allowed to cite a result from the homework, though try to be specific in your response.)

False. As shown on Homework 4.10, if $X_1, \ldots, X_n$ is a random sample of size n from the uniform distribution on the three points $\{\theta, \theta+1, \theta+2\}$, where $\theta \in \mathbf{Z}$, then a complete sufficient statistic does not exist.

(c) A constant function is both ancillary and complete.

True. Holds by definition. A constant function Y is independent of the parameter θ. Also, if $f\colon \mathbf{R} \to \mathbf{R}$ satisfies $\mathbf{E}_\theta f(Y) = 0$, then since Y is constant, $f(Y)$ is also constant, so we have $0 = \mathbf{E}_\theta f(Y) = f(Y)$, so that Y is complete.

(d) A complete sufficient statistic is minimal sufficient.

True (under the assumptions of Bahadur's Theorem). Bahadur's Theorem, from the notes, Theorem 5.25, says this is true under the assumption that either $\{f_\theta \colon \theta \in \Theta\}$ is a family of probability density functions, or it is a family of probability mass functions such that the set $\cup_{\theta\in\Theta}\{x \in \mathbf{R}^n \colon f_\theta(x) > 0\}$ is countable. These assumptions were needed to get the existence of a minimal sufficient statistic. However, a minimal sufficient statistic might not exist in general (as shown by Bahadur in 1954), in which case the argument to prove statement (d) breaks down. [2]

(e) Let $X_1, \ldots, X_n$ be a random sample of size n from the Gaussian distribution with mean $\mu \in \mathbf{R}$ and variance $\sigma = 1$, so that μ is unknown. Let Y be a complete sufficient statistic for μ. Let Z be an ancillary statistic for μ. Then Y and Z are independent.

True (under the assumptions of Basu's Theorem). This follows from Basu's Theorem, from the notes, Theorem 5.27. However, we technically did not state under which probability distribution the random variables are independent, but that is implicit I suppose.

2. Question 2

Give an example of a statistic Y that is complete and nonconstant, but such that Y is not sufficient.

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[2]As discussed in Landers-Rogge 1972, the counterexample goes like this. For any set A, let 2^A denote the set of subsets of A. Let $\Omega := \prod_{t\in\mathbf{R}}\Omega_t$, where $\Omega_t = \{0,1\}$ for all $t \in \mathbf{R}$. Let P, Q be probability measures on $\{0,1\}$ such that $P(1) = 1$ and $Q(0) = Q(1) = 1/2$. For any $t \in \mathbf{R}$, let $\mathbf{P}_t$ be the product probability measure on Ω such that $\mathbf{P}_t$ restricted to Ω_t is P, and $\mathbf{P}_t$ restricted to any other product term Ω_s with $s \neq t$ is Q. For any $t \in \mathbf{R}$ define $h_t \colon \Omega \to \mathbf{R}$ so that $h_t(\omega) = 2 \cdot 1_{\omega_t}$, for all $\omega = (\omega_t)_{t\in\mathbf{R}} \in \Omega$. Then the family of functions $\{h_t\}_{t\in\mathbf{R}}$ are probability densities with respect to the infinite product measure $\prod_{t\in\mathbf{R}} Q$.

Solution. There are many examples. Here is one. Let $X_1, \ldots, X_n$ be Bernoulli random variables with unknown parameter $0 < \theta < 1$. Define $t(x_1, \ldots, x_n) := x_1$. We claim that $Y := t(X_1, \ldots, X_n)$ is complete but not sufficient. Completeness follows since if $f\colon \mathbf{R} \to \mathbf{R}$ satisfies $\mathbf{E}_\theta f(Y) = 0$ for all $0 < \theta < 1$, then $\theta f(1) + (1-\theta) f(0)$ for all $0 < \theta < 1$, so that $f(0) = f(1) = 0$, so that $f(Y) = 0$, i.e. Y is complete. However, Y is not sufficient since $X_1, \ldots, X_n$ conditioned on X_1 does depend on θ. For example,

$$\mathbf{P}(X_1 = 1, X_2 = 1 | X_1 = 1) = \mathbf{P}(X_2 = 1) = \theta.$$

3. Question 3

Suppose $X_1, \ldots, X_n$ is a random sample of size n from the Gaussian distribution with unknown mean $\mu \in \mathbf{R}$ and unknown variance $\sigma^2 > 0$. You may freely use that the sample mean $\overline{X}$ is UMVU for μ and $(\overline{X}, S^2)$ is complete sufficient for (μ, σ^2), where $S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$.

Fix $\sigma^2 > 0$. Give an explicit expression for a statistic Y that is UMVU for μ^2, and prove that Y is UMVU for μ^2.

Solution. Note that $\mathbf{E}\overline{X}^2 = \mu^2 + \sigma^2/n$, so

$$\mathbf{E}[\overline{X}^2 - S^2/n] = \mu^2.$$

We claim that $Y := \overline{X}^2 - S^2/n$ is UMVU for μ^2 (with fixed σ). To see this, note that $\overline{X}^2 - S^2/n$ is a function of the complete sufficient statistic $Z = (\overline{X}, S^2)$, using $h(x, y) := x - y/n$ and $g(\mu, \sigma^2) := \mu^2$. Let $\theta = (\mu, \sigma^2)$. By the Lehman-Scheffé Theorem (Theorem 6.13 in the notes), $W := \mathbf{E}_\theta(Y|Z)$ is UMVU for $g(\theta) = \mu^2$. Since Y is already a function of Z, we have $\mathbf{E}_\theta(Y|Z) = Y$, by Exercise 5.1(iii). So, Y is UMVU for μ^2.

4. Question 4

Prove the Rao-Blackwell Theorem:

Let Z be a sufficient statistic for $\{f_\theta \colon \theta \in \Theta\}$ and let Y be an estimator for $g(\theta)$. Define $W := \mathbf{E}_\theta(Y|Z)$. Let $\theta \in \Theta$ with $r(\theta, Y) < \infty$ and such that $\ell(\theta, y)$ is convex in $y \in \mathbf{R}$. Then

$$r(\theta, W) \leq r(\theta, Y).$$

(Recall that $\ell\colon \Theta \times \mathbf{R}^k \to \mathbf{R}$, and $r(\theta, Y) := \mathbf{E}_\theta \ell(\theta, Y)$.)

Solution. By the (conditional) Jensen's inequality,

$$\ell(\theta, W) = \ell(\theta, \mathbf{E}_\theta(Y|Z)) \leq \mathbf{E}_\theta[\ell(\theta, Y)|Z].$$

Taking expected values of both sides and applying the total Expectation Theorem for conditional expectation, we get

$$r(\theta, W) \leq \mathbf{E}_\theta \mathbf{E}_\theta[\ell(\theta, Y)|Z] = \mathbf{E}_\theta \ell(\theta, Y) = r(\theta, Y).$$

And if $\ell(\theta, y)$ is strictly convex in y, then this inequality is strict, unless Y is a function of Z. If Y is a function of Z, then $\mathbf{E}_\theta(Y|Z) = Y$, so $W = Y$.