541B Midterm 1 Solutions[1]

1. Question 1

Let $X_1, \ldots, X_n$ be a random sample of size n from a Poisson distribution with unknown parameter $\lambda > 0$. (So, $\mathbf{P}(X_1 = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$.)

- Find an MLE (maximum likelihood estimator) for λ.
- Is the MLE you found unique? That is, could there be more than one MLE for this problem?

Solution. The MLE of θ is a value of θ maximizing

$$\log \prod_{i=1}^{n} \theta^{X_i} e^{-\theta}/X_i! = \log\Big(\theta^{\sum_{i=1}^n X_i} e^{-n\theta} \prod_{i=1}^{n} [X_i!]\Big) = \sum_{i=1}^{n} \log(X_i!) - n\theta + \log\theta \sum_{i=1}^{n} X_i.$$

Taking a derivative in θ, we get $-n + \frac{1}{\theta}\sum_{i=1}^n X_i$. From the first derivative test, there is a unique maximum value of θ when $\theta = \frac{1}{n}\sum_{i=1}^n X_i$, so the MLE for θ is $\frac{1}{n}\sum_{i=1}^n X_i$.

2. Question 2

Suppose X is a binomial distributed random variable with parameters 2 and $\theta \in \{1/2, 3/4\}$. (That is, X is the number of heads that result from flipping two coins, where each coin has probability θ of landing heads.)
We want to test the hypothesis H_0 that $\theta = 1/2$ versus the hypothesis H_1 that $\theta = 3/4$.

Let $\mathcal{T}$ be the set of hypothesis tests with significance level at most 1/40.
(Recall that the significance level of a hypothesis test $\phi\colon \mathbf{R} \to [0,1]$ is $\sup_{\theta\in\Theta_0} \mathbf{E}_\theta \phi(X)$.)

Find a uniformly most powerful (UMP) class $\mathcal{T}$ hypothesis test $\phi\colon \mathbf{R} \to [0,1]$.
Compute all constants that appear in the definition of ϕ. Justify your answer.

Hint: you can freely use the following facts about the PMF f_θ of X

$$\frac{f_{3/4}(0)}{f_{1/2}(0)} = \frac{1}{4}, \qquad \frac{f_{3/4}(1)}{f_{1/2}(1)} = \frac{3}{4}, \qquad \frac{f_{3/4}(2)}{f_{1/2}(2)} = \frac{9}{4}.$$

Solution. The Neyman-Pearson Lemma says that a UMP test for the class of tests with an upper bound on the significance level must be a likelihood ratio test with significance level equal to 1/40. That is, there is some $k > 0$ and $\gamma \in [0,1]$ such that the following hypothesis test is UMP class $\mathcal{T}$.

$$\phi(x) := \begin{cases} 1 & , \text{if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & , \text{if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ \gamma & , \text{if } f_{\theta_1}(x) = k f_{\theta_0}(x). \end{cases}$$

After examining the likelihood ratios, we decide to choose $k = 9/4$, so that

$$\phi(x) := \begin{cases} 1 & , \text{if } f_{\theta_1}(x) > (9/4) f_{\theta_0}(x) \\ 0 & , \text{if } f_{\theta_1}(x) < (9/4) f_{\theta_0}(x) \\ \gamma & , \text{if } f_{\theta_1}(x) = (9/4) f_{\theta_0}(x) \end{cases} = \begin{cases} 0 & , \text{if } x \neq 2 \\ \gamma & , \text{if } x = 2. \end{cases}$$

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Then $\mathbf{E}_{\theta_0}\phi(X) = \mathbf{P}_{\theta_0}(X = 2)\gamma = \mathbf{P}_{1/2}(X = 2)\gamma = (1/4)\gamma$. Since this quantity is equal to $1/40$ by assumption, we choose $\gamma := 1/10$. That is, our UMP test is

$$\phi(x) := \begin{cases} 0 & \text{, if } x \neq 2 \\ 1/10 & \text{, if } x = 2. \end{cases}$$

3. Question 3

Let $X_1, \ldots, X_n$ be a real-valued random sample of size n so that X_1 has PDF given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}, \qquad \forall\, x \in \mathbf{R},$$

where $\mu \in \mathbf{R}$ is an unknown parameter.

Suppose we want to test the hypothesis H_0 that $\mu = 0$ versus the hypothesis H_1 that $\mu > 0$.

- Describe a uniformly most powerful hypothesis test among all hypothesis tests with significance level at most $1/3$. Justify your answer.
- Write a p-value for the test you found. Simplify the expression to a reasonable extent.

Solution. We have for any $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$,

$$f_\mu(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\mu)^2/2} = (2\pi)^{-n/2} e^{-\sum_{i=1}^{n}(x_i-\mu)^2}.$$

So, if $\mu_1 > \mu_0 > 0$, we have for any $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$,

$$\frac{f_{\mu_1}(x)}{f_{\mu_0}(x)} = e^{-\sum_{i=1}^{n}[(x_i-\mu_1)^2-(x_i-\mu_0)^2]/2} = e^{\sum_{i=1}^{n}[2x_i(\mu_1-\mu_0)-(\mu_1^2-\mu_0^2)/2]}.$$

Since $\mu_1 - \mu_0 > 0$, this likelihood ratio is a strictly increasing function of $\sum_{i=1}^{n} x_i$. We conclude from the Karlin-Rubin Theorem (with $\Theta = [0, \infty)$) that there is a UMP test of the form

$$\phi(x) := \begin{cases} 1 & \text{, if } \sum_{i=1}^{n} x_i > c \\ 0 & \text{, if } \sum_{i=1}^{n} x_i < c \\ \gamma & \text{, if } \sum_{i=1}^{n} x_i = c. \end{cases}$$

for some $c \in \mathbf{R}$. Rewriting this, and noting that $\sum_{i=1}^{n} x_i = c$ has probability zero for any $\mu \in \mathbf{R}$,

$$\phi(x) := \begin{cases} 1 & \text{, if } \sum_{i=1}^{n} x_i < c \\ 0 & \text{, if } \sum_{i=1}^{n} x_i \geq c. \end{cases}$$

A p-value is then $p(X)$ where $X = (X_1, \ldots, X_n)$ and

$$p(x) := \mathbf{P}_0\Big(\sum_{i=1}^{n} X_i \geq \sum_{i=1}^{n} x_i\Big), \qquad \forall\, x = (x_1, \ldots, x_n) \in \mathbf{R}^n.$$

Since $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ is a mean 0 Gaussian with variance 1, we could rewrite this as

$$p(x) = \int_{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

4. Question 4

Suppose $\Theta = \{\theta_0, \theta_1\}$, $\Theta_0 = \{\theta_0\}$, $\Theta_1 = \{\theta_1\}$. Let H_0 be the hypothesis $\{\theta = \theta_0\}$ and let H_1 be the hypothesis $\{\theta = \theta_1\}$. Let $\{f_{\theta_0}, f_{\theta_1}\}$ be two multivariable probability densities on $\mathbf{R}^n$. Fix $k \geq 0$. Define a **likelihood ratio test** $\phi\colon \mathbf{R}^n \to [0,1]$ to be

$$\phi(x) := \begin{cases} 1 & \text{, if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & \text{, if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ \text{(unspecified)} & \text{, if } f_{\theta_1}(x) = k f_{\theta_0}(x). \end{cases} \qquad (*)$$

Define

$$\alpha := \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0) = \mathbf{E}_{\theta_0} \phi(X). \qquad (**)$$

Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most α.

Prove the following:

Any randomized hypothesis test satisfying $(*)$ is a UMP class $\mathcal{T}$ test.

Solution.

As we already noted in $(**)$, Θ_0 consists of a single point, so the supremum appearing in $(**)$ is just $\beta(\theta_0)$, and we will repeatedly use this fact below without further mention.

Let $\beta(\theta)$ be the power function of the test corresponding to ϕ. Let ϕ' another test in $\mathcal{T}$, and let $\beta'(\theta)$ be the power function of this test. By definition of ϕ, we have

$$[\phi(x) - \phi'(x)][f_{\theta_1}(x) - k f_{\theta_0}(x)] \geq 0, \qquad \forall\, x \in \mathbf{R}^n.$$

Therefore,

$$0 \leq \int_{\mathbf{R}^n} [\phi(x) - \phi'(x)][f_{\theta_1}(x) - k f_{\theta_0}(x)] dx = \beta(\theta_1) - \beta'(\theta_1) - k[\beta(\theta_0) - \beta'(\theta_0)]. \qquad (***)$$

Since ϕ has significance level α and ϕ' has significance level at most α, we have $\beta(\theta_0) - \beta'(\theta_0) \geq 0$. So, $k \geq 0$ and $(***)$ imply that $\beta(\theta_1) - \beta'(\theta_1) \geq 0$. That is, the ϕ test is UMP class $\mathcal{T}$.