Math 541B, Fall 2025, USC Instructor: Steven Heilman

Name: ______________________ USC ID: ______________ Date: ___________

Signature: ______________________. Discussion Section: ______________________
(By signing here, I certify that I have taken this test while refraining from cheating.)

# Exam 1

This exam contains 6 pages (including this cover page) and 4 problems. Enter all requested information on the top of this page.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| Total: | 40 | |

Do not write in the table to the right. Good luck![a]

1. (10 points) Let $X_1, \ldots, X_n$ be a random sample of size $n$ from a Poisson distribution with unknown parameter $\lambda > 0$. (So, $\mathbf{P}(X_1 = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$.)

   - Find an MLE (maximum likelihood estimator) for $\lambda$.
   - Is the MLE you found unique? That is, could there be more than one MLE for this problem?

   [This was a repeated homework question.]

2. (10 points) Suppose $X$ is a binomial distributed random variable with parameters 2 and $\theta \in \{1/2, 3/4\}$. (That is, $X$ is the number of heads that result from flipping two coins, where each coin has probability $\theta$ of landing heads.)
We want to test the hypothesis $H_0$ that $\theta = 1/2$ versus the hypothesis $H_1$ that $\theta = 3/4$.

Let $\mathcal{T}$ be the set of hypothesis tests with significance level at most $1/40$.
(Recall that the significance level of a hypothesis test $\phi\colon \mathbf{R} \to [0,1]$ is $\sup_{\theta \in \Theta_0} \mathbf{E}_\theta \phi(X)$.)

Find a uniformly most powerful (UMP) class $\mathcal{T}$ hypothesis test $\phi\colon \mathbf{R} \to [0,1]$.
Compute all constants that appear in the definition of $\phi$. Justify your answer.

Hint: you can freely use the following facts about the PMF $f_\theta$ of $X$

$$\frac{f_{3/4}(0)}{f_{1/2}(0)} = \frac{1}{4}, \qquad \frac{f_{3/4}(1)}{f_{1/2}(1)} = \frac{3}{4}, \qquad \frac{f_{3/4}(2)}{f_{1/2}(2)} = \frac{9}{4}.$$

[This was a repeated homework question /practice exam question.]

3. (10 points) Let $X_1, \ldots, X_n$ be a real-valued random sample of size $n$ so that $X_1$ has PDF given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}, \qquad \forall\, x \in \mathbf{R},$$

where $\mu \in \mathbf{R}$ is an unknown parameter.

Suppose we want to test the hypothesis $H_0$ that $\mu = 0$ versus the hypothesis $H_1$ that $\mu > 0$.

- Describe a uniformly most powerful hypothesis test among all hypothesis tests with significance level at most 1/3. Justify your answer.
- Write a $p$-value for the test you found. Simplify the expression to a reasonable extent.

[This was a modified homework/practice exam question.]

4. (10 points) Suppose $\Theta = \{\theta_0, \theta_1\}$, $\Theta_0 = \{\theta_0\}$, $\Theta_1 = \{\theta_1\}$. Let $H_0$ be the hypothesis $\{\theta = \theta_0\}$ and let $H_1$ be the hypothesis $\{\theta = \theta_1\}$. Let $\{f_{\theta_0}, f_{\theta_1}\}$ be two multivariable probability densities on $\mathbf{R}^n$. Fix $k \geq 0$. Define a **likelihood ratio test** $\phi\colon \mathbf{R}^n \to [0,1]$ to be

$$\phi(x) := \begin{cases} 1 & \text{, if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & \text{, if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ \text{(unspecified)} & \text{, if } f_{\theta_1}(x) = k f_{\theta_0}(x). \end{cases} \qquad (*)$$

Define

$$\alpha := \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0) = \mathbf{E}_{\theta_0} \phi(X). \qquad (**)$$

Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most $\alpha$.

Prove the following:

Any randomized hypothesis test satisfying $(*)$ is a UMP class $\mathcal{T}$ test.

[This exercise asks you to prove the first part of the Neyman-Pearson Lemma.]

(Scratch paper)