Please provide complete and well-written solutions to the following exercises.

Due September 19, 6PM, to be submitted in brightspace, under the Content tab (so it actually is submitted to gradescope).

Homework 2

Exercise 1. The rejection regions C_α for UMP hypothesis tests of significance level at most $\alpha \in (0,1)$ are often nested in the sense that $C_\alpha \subseteq C_{\alpha'}$ for all $0 < \alpha < \alpha' < 1$. This exercise demonstrates an example of UMP tests where this nesting behavior does not occur.

Let $\theta_0, \theta_1 \in \mathbf{R}$ be unequal parameters. Let H_0 denote the hypothesis $\{\theta = \theta_0\}$ and let H_1 denote the hypothesis $\{\theta = \theta_1\}$. Suppose $X \in \{1,2,3\}$ is a random variable. If $\theta = \theta_0$, assume that X takes the values $1,2,3$ with probabilities $.85, .1, .05$, respectively. If $\theta = \theta_1$, assume that X takes the values $1,2,3$ with probabilities $.7, .2, .1$, respectively. Let $\mathcal{T}$ denote the set of hypothesis tests with significance level at most α.

- Let $0 < \alpha < .15$. Show that a UMP class $\mathcal{T}$ test is not unique.
- When $\alpha = .05$, show there is a unique nonrandomized hypothesis UMP class $\mathcal{T}$ test.
- When $\alpha = .1$, show there is a unique nonrandomized hypothesis UMP class $\mathcal{T}$ test.
- Show that the $\alpha = .05$ and $\alpha' = .1$ UMP nonrandomized tests from above do not have nested rejection regions.
- However, when $\alpha = .05$ and $\alpha' = .1$, there are randomized UMP tests $\phi, \phi' \colon \mathbf{R}^n \to [0,1]$ respectively, that are nested in the sense that $\phi \leq \phi'$.

Exercise 2. Suppose X is a Gaussian distributed random variable with known variance $\sigma^2 > 0$ but unknown mean. Fix $\mu_0, \mu_1 \in \mathbf{R}$. Assume that $\mu_0 - \mu_1 > 0$. We want to test the hypothesis H_0 that $\mu = \mu_0$ versus the hypothesis H_1 that $\mu = \mu_1$. Fix $\alpha \in (0,1)$. Explicitly describe the UMP test for the class of tests whose significance level is at most α.

Your description of the test should use the function $\Phi(t) := \int_{-\infty}^{t} e^{-x^2/2} dx/\sqrt{2\pi}$, $\Phi \colon \mathbf{R} \to (0,1)$, and/or the function $\Phi^{-1} \colon (0,1) \to \mathbf{R}$. (Recall that $\Phi(\Phi^{-1}(s)) = s$ for all $s \in (0,1)$ and $\Phi^{-1}(\Phi(t)) = t$ for all $t \in \mathbf{R}$.)

Exercise 3. This exercise demonstrates that a UMP might not always exists.

Let $X_1, \ldots, X_n$ be i.i.d. Gaussian random variables with known variance and unknown mean $\mu \in \mathbf{R}$. Fix $\mu_0 \in \mathbf{R}$. Let H_0 denote the hypothesis $\{\mu = \mu_0\}$ and let H_1 denote the hypothesis $\mu \neq \mu_0$. Fix $0 < \alpha < 1$. Let $\mathcal{T}$ denote the set of hypothesis tests with significance level at most α. Show that no UMP class $\mathcal{T}$ test exists, using the following strategy.

- Let $\mu_1 < \mu_0$. You may take as given the following fact (that follows from the Karlin-Rubin Theorem): the power at μ_1 is maximized among class $\mathcal{T}$ tests by the hypothesis test ϕ that rejects H_0 when the sample mean satisfies $\overline{X} < c$ for an appropriate choice of $c \in \mathbf{R}$. Assume for the sake of contradiction that a UMP class $\mathcal{T}$ test ϕ' exists. Then, using the necessity part of the Neyman-Pearson Lemma (i.e. consider testing $\mu = \mu_0$ versus $\mu = \mu_1$), conclude that ϕ' must have the same rejection region as ϕ (just by examining the power of the tests at μ_1.)

- Consider now a test in $\mathcal{T}$ that rejects H_0 when the sample mean satisfies $\overline{X} > c'$ for an appropriate choice of $c' \in \mathbf{R}$. Repeating the previous argument, conclude that ϕ' must reject when $\overline{X} > c'$, leading to a contradiction.

  That is, let $\mu_2 > \mu_0$. You may take as given the following fact (that follows from the Karlin-Rubin Theorem): the power at μ_2 is maximized among class $\mathcal{T}$ tests by the hypothesis test ϕ'' that rejects H_0 when the sample mean satisfies $\overline{X} > c'$ for an appropriate choice of $c' \in \mathbf{R}$. Then, using the necessity part of the Neyman-Pearson Lemma (i.e. consider testing $\mu = \mu_0$ versus $\mu = \mu_2$), conclude that ϕ' must have the same rejection region as ϕ''.

Exercise 4. Prove the following version of the Karlin-Rubin Theorem, with the inequalities reversed in the definition of the hypotheses.

Let $\{f_\theta\}$ be a family of PDFs with the MLR property, with respect to a real-valued statistic $Y = t(X)$, where $\theta \in \Theta \subseteq \mathbf{R}$. Let $0 \leq \gamma \leq 1$. Fix $\theta_0 \in \Theta$. Consider the hypothesis $H_0 = \{\theta \geq \theta_0\}$ and the hypothesis $H_1 = \{\theta < \theta_0\}$. Let $c \in \mathbf{R}$. Consider the randomized hypothesis test $\phi\colon \mathbf{R}^n \to [0,1]$ defined by

$$\phi(x) := \begin{cases} 0 & , \text{if } t(x) > c \\ 1 & , \text{if } t(x) < c \\ \gamma & , \text{if } t(x) = c. \end{cases}$$

Define $\alpha := \mathbf{E}_{\theta_0}\phi(X)$. Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most α.

(i) ϕ is UMP class $\mathcal{T}$.

(iii) β, the power function of ϕ, is nonincreasing and strictly decreasing when it takes values in $(0,1)$.

Exercise 5. Prove the following one-sided version of the Karlin-Rubin Theorem.

Let $\{f_\theta\}$ be a family of PDFs with the MLR property, with respect to a real-valued statistic $Y = t(X)$, where $\theta \in \Theta \subseteq \mathbf{R}$. Let $0 \leq \gamma \leq 1$. Fix $\theta_0 \in \Theta$. Consider the hypothesis $H_0 = \{\theta = \theta_0\}$ and the hypothesis $H_1 = \{\theta > \theta_0\}$. Let $c \in \mathbf{R}$. Consider the randomized hypothesis test $\phi\colon \mathbf{R}^n \to [0,1]$ defined by

$$\phi(x) := \begin{cases} 1 & , \text{if } t(x) > c \\ 0 & , \text{if } t(x) < c \\ \gamma & , \text{if } t(x) = c. \end{cases}$$

Define $\alpha := \mathbf{E}_{\theta_0}\phi(X)$. Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most α.

Then ϕ is UMP class $\mathcal{T}$.

Exercise 6. Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $X = (X_1, \ldots, X_n)$. Let $\theta > 0$. Assume that X_1 is uniformly distributed in the interval $[0, \theta]$. Fix $\theta_0 > 0$. Fix $0 < \alpha < 1$. Let $\mathcal{T}$ denote the set of hypothesis tests with significance level at most α.

- Suppose we test $H_0 = \{\theta \leq \theta_0\}$ versus $H_1 = \{\theta > \theta_0\}$. Identify the set of all UMP class $\mathcal{T}$ hypothesis tests.
- Suppose we test $H_0 = \{\theta = \theta_0\}$ versus $H_1 = \{\theta \neq \theta_0\}$. Show there is a unique UMP class $\mathcal{T}$ hypothesis test in this case.

(Hint: first consider testing $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$ with $\theta_1 > \theta_0$, and apply the Neyman-Pearson Lemma. That is, mimic the argument of the Karlin-Rubin Theorem.) (As an aside, observe that, if you naïvely apply the Karlin-Rubin Theorem, you will not find all UMP tests, i.e. a non-strict MLR property version of the Karlin-Rubin Theorem will neglect some UMP tests.)

Exercise 7. Let $X_1, \ldots, X_n$ be i.i.d. random variables that are uniformly distributed in the interval $[\theta, \theta + 1]$, where $\theta \in \mathbf{R}$ is an unknown parameter. Fix $\theta_0 \in \mathbf{R}$. Suppose we want to test the hypothesis that $\theta \leq \theta_0$ versus $\theta > \theta_0$. For any $0 \leq \alpha \leq 1$, show that there exists a UMP test among tests with significance level at most α, and this test rejects the null hypothesis when $X_{(1)} > \theta_0 + c(\alpha)$ or $X_{(n)} > \theta_0 + 1$.

On the other hand, show that the joint density of $X_1, \ldots, X_n$ does not have the MLR property with respect to any statistic (when $n > 1$). (Hint: if it did have the MLR property, what would the Karlin-Rubin Theorem imply about the UMP rejection regions?)

(Hint: for the first part, first consider testing $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$ with $\theta_1 > \theta_0$, and apply the Neyman-Pearson Lemma.)

Exercise 8. Let $\{f_\theta \colon \theta \in \mathbf{R}\}$ be a family of positive, single-variable PDFs, i.e. $f_\theta \colon \mathbf{R} \to (0, \infty)$ for all $\theta \in \mathbf{R}$. Assume that $f_\theta(x)$ is twice continuously differentiable in the parameters θ, x.

Show that $\{f_\theta\}$ has the MLR property with respect to the statistic $t(x) = x$ ($x \in \mathbf{R}$) if and only if

$$\frac{\partial^2}{\partial\theta\partial x} \log f_\theta(x) \geq 0, \qquad \forall\, x, \theta \in \mathbf{R}.$$

Exercise 9. Suppose X is a binomial distributed random variable with parameters $n = 100$ and $\theta \in [0, 1]$ where θ is unknown. Suppose we want to test the hypothesis H_0 that $\theta = 1/2$ versus the hypothesis H_1 that $\theta \neq 1/2$. Consider the hypothesis test that rejects the null hypothesis if and only if $|X - 50| > 10$.

Using e.g. the central limit theorem, do the following:

- Give an approximation to the significance level α of this hypothesis test
- Plot an approximation of the power function $\beta(\theta)$ as a function of θ.
- (Optional) Estimate p values for this test when $X = 50$, and also when $X = 70$ or $X = 90$.

Exercise 10. Let $X_1, \ldots, X_n$ be a real-valued random sample of size n from a family of distributions $\{f_\theta \colon \theta \in \Theta\}$. Suppose $\Theta = \mathbf{R}$. Fix $\theta \in \mathbf{R}$. Denote $X := (X_1, \ldots, X_n)$. Consider a set of hypothesis tests $\phi_\alpha \colon \mathbf{R}^n \to [0,1]$, for any $\alpha \in [0,1]$. Assume that these tests are nested in the sense that $\phi_\alpha \leq \phi_{\alpha'}$ for all $0 \leq \alpha < \alpha' \leq 1$. Suppose we are testing the hypothesis H_0 that $\{\theta \leq \theta_0\}$ versus H_1 that $\{\theta > \theta_0\}$. Suppose also that $\{f_\theta\}$ has the monotone likelihood ratio property with respect to a statistic $Y = t(X)$ that is a continuous random variable.

- Show that the family of UMP tests with significance level at most α satisfies the nested property mentioned above (for all $\alpha \in [0,1]$).
- (Optional) Show that, if $X = x$, then the p-value $p(x)$ satisfies

  $$p(x) = \mathbf{P}_{\theta_0}(t(X) > t(x)).$$

  (A p-value is defined for randomized tests as $p(X) := \inf\{\alpha \in [0,1] \colon \phi_\alpha(X) = 1\}$.)