MATH 541B, GRADUATE STATISTICS
SELECTED HOMEWORK SOLUTIONS

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Contents

1. Homework 1

Exercise 1.1. Estimate the probability that 1000000 coin flips of fair coins will result in more than 501,000 heads, using the Central Limit Theorem. (Some of the following integrals may be relevant: $\int_{-\infty}^{0} e^{-t^2/2}dt/\sqrt{2\pi} = 1/2$, $\int_{-\infty}^{1} e^{-t^2/2}dt/\sqrt{2\pi} \approx .8413$, $\int_{-\infty}^{2} e^{-t^2/2}dt/\sqrt{2\pi} \approx .9772$, $\int_{-\infty}^{3} e^{-t^2/2}dt/\sqrt{2\pi} \approx .9987$.) (Hint: use Bernoulli random variables.)

Casinos do these kinds of calculations to make sure they make money and that they do not go bankrupt. Financial institutions and insurance companies do similar calculations for similar reasons.

Solution. For any $1 \leq 1$, let $X_i = 1$ if the i^{th} coin flip is heads and $X_i = 0$ otherwise. We assume that $X_1, \ldots$ are iid with $\mathbf{P}(X_1 = 1) = 1/2$, $\mathbf{E}X_1 = 1/2$ and $\mathrm{var}(X_1) = 1/4$. We want to know the probability that

$$X_1 + \cdots + X_{10^7} > 501000.$$

Equivalently, we want the probability of the event

$$\{X_1 + \cdots + X_{10^7} - 10^7/2 > 1000\} = \{\frac{X_1 + \cdots + X_{10^7} - 10^7/2}{\sqrt{10^6}\sqrt{1/4}} > 2\} =$$

Using the Central Limit Theorem as an approximation, we have the approximation

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_{10^7} - 10^7/2}{\sqrt{10^6}\sqrt{1/4}} > 2\right) \approx \int_2^\infty e^{-x^2/2}dx/\sqrt{2\pi}$$
$$= 1 - \int_\infty^2 e^{-x^2/2}dx/\sqrt{2\pi} \approx 1 - .9772 = .0228.$$

□

Exercise 1.2 (Numerical Integration). In computer graphics in video games, etc., various integrations are performed in order to simulate lighting effects. Here is a way to use random sampling to integrate a function in order to quickly and accurately render lighting effects.

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Let $\Omega = [0, 1]$, and let $\mathbf{P}$ be the uniform probably law on Ω, so that if $0 \leq a < b \leq 1$, we have $\mathbf{P}([a, b]) = b - a$. Let $X_1, \ldots, X_n$ be independent random variables such that $\mathbf{P}(X_i \in [a, b]) = b - a$ for all $0 \leq a < b \leq 1$, for all $i \in \{1, \ldots, n\}$. Let $f\colon [0, 1] \to \mathbb{R}$ be a continuous function we would like to integrate. Instead of integrating f directly, we instead compute the quantity

$$\frac{1}{n} \sum_{i=1}^{n} f(X_i).$$

Show that

$$\lim_{n \to \infty} \mathbf{E} \left(\frac{1}{n} \sum_{i=1}^{n} f(X_i) \right) = \int_0^1 f(t) dt.$$

$$\lim_{n \to \infty} \text{var} \left(\frac{1}{n} \sum_{i=1}^{n} f(X_i) \right) = 0.$$

That is, as n becomes large, $\frac{1}{n} \sum_{i=1}^{n} f(X_i)$ is a good estimate for $\int_0^1 f(t) dt$.

Solution. By definition of X_i we have $\mathbf{E} f(X_i) = \int_0^1 f(t) dt$ for all $i \geq 1$so that $\mathbf{E} \left(\frac{1}{n} \sum_{i=1}^{n} f(X_i) \right) = \frac{1}{n} n \int_0^1 f(t) dt = \int_0^1 f(t) dt$. Also, by independence we have

$$\text{var} \left(\frac{1}{n} \sum_{i=1}^{n} f(X_i) \right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var}(f(X_i)) = \frac{1}{n} \text{var}(f(X_1)).$$

This quantity goes to zero as $n \to \infty$. (Since f is continuous on $[0, 1]$, f is bounded by some constant c on $[0, 1]$, i.e. $|f(t)| \leq c$ for all $t \in [0, 1]$, so $|f(X_1)| \leq c$, so $\text{var} f(X_i) \leq \mathbf{E}[f(X_i)]^2 \leq c^2$ for all $i \geq 1$.) $\square$

Exercise 1.3. Let $X := (X_1, \ldots, X_n)$ be a random sample of size n from a binomial distribution with parameters n and p. Here n is a positive (known) integer and $0 < p < 1$ is unknown. (That is, $X_1, \ldots, X_n$ are i.i.d. and X_1 is a binomial random variable with parameters n and p, so that $\mathbf{P}(X_1 = k) = \binom{n}{k} p^k (1 - p)^{n-k}$ for all integers $0 \leq k \leq n$.)

You can freely use that $\mathbf{E} X_1 = np$ and $\text{Var} X_1 = np(1 - p)$.

- Computer the Fisher information $I_X(p)$ for any $0 < p < 1$. (Consider n to be fixed.)
- Let Z be an unbiased estimator of p^2 (assume that Z is a function of $X_1, \ldots, X_n$). State the Cramér-Rao inequality for Z.
- Let W be an unbiased estimator of $1/p$ (assume that W is a function of $X_1, \ldots, X_n$). State the Cramér-Rao inequality for W.

Solution. Using that the information of independent random variables is the sum of the informations, using the alternate definition of Fisher information using the variance, and using that the variance is unchanged by adding a constant inside the variance,

$$
\begin{aligned}
I_X(p) = nI_{X_1}(p) &= n\mathrm{Var}_p\Big(\frac{d}{dp}\Big[\log\Big(\binom{n}{X_1}p^{X_1}(1-p)^{n-X_1}\Big)\Big]\Big)\\
&= n\mathrm{Var}_p\Big(\frac{d}{dp}\Big[\log\binom{n}{X_1} + X_1\log p + (n-X_1)\log(1-p)\Big]\Big)\\
&= n\mathrm{Var}_p\Big(\frac{d}{dp}\Big[X_1\log p + (n-X_1)\log(1-p)\Big]\Big)\\
&= n\mathrm{Var}_p\Big(\frac{1}{p}X_1 - \frac{1}{1-p}(n-X_1)\Big) = n\mathrm{Var}_p\Big(\Big[\frac{1}{p}+\frac{1}{1-p}\Big]X_1\Big)\\
&= n\Big[\frac{1}{p}+\frac{1}{1-p}\Big]^2\mathrm{Var}_p X_1 = n\Big[\frac{1}{p(1-p)}\Big]^2 np(1-p) = \frac{n^2}{p(1-p)}
\end{aligned}
$$

The Cramér-Rao inequality says, if $g(p) := \mathbf{E}_p Z$, then

$$\mathrm{Var}_p(Z) \geq \frac{|g'(p)|^2}{I_X(p)}.$$

If $g(p) = p^2$, then $g'(p) = 2p$, so we get

$$\mathrm{Var}_p(Z) \geq \frac{(2p)^2}{I_X(p)} = \frac{4p^3(1-p)}{n^2}.$$

If $g(p) = 1/p$, then $g'(p) = -p^{-2}$, so we get

$$\mathrm{Var}_p(Z) \geq \frac{p^{-4}}{I_X(p)} = p^{-3}\frac{1-p}{n^2}.$$

□

Exercise 1.4. Let $X_1, \ldots, X_n$ be a random sample of size n from a Poisson distribution with unknown parameter $\lambda > 0$. (So, $\mathbf{P}(X_1 = k) = e^{-\lambda}\lambda^k/k!$ for all integers $k \geq 0$.)

- Find an MLE (maximum likelihood estimator) for λ.
- Is the MLE you found unique? That is, could there be more than one MLE for this problem?

Solution. The MLE of θ is a value of θ maximizing

$$\log\prod_{i=1}^n \theta^{X_i}e^{-\theta}/X_i! = \log\Big(\theta^{\sum_{i=1}^n X_i}e^{-n\theta}\prod_{i=1}^n[X_i!]\Big) = \sum_{i=1}^n \log(X_i!) - n\theta + \log\theta\sum_{i=1}^n X_i.$$

Taking a derivative in θ, we get $-n + \frac{1}{\theta}\sum_{i=1}^n X_i$. From the first derivative test, there is a unique maximum value of θ when $\theta = \frac{1}{n}\sum_{i=1}^n X_i$, so the MLE for θ is $\frac{1}{n}\sum_{i=1}^n X_i$. □

2. Homework 2

Exercise 2.1. The rejection regions C_α for UMP hypothesis tests of significance level at most $\alpha \in (0,1)$ are often nested in the sense that $C_\alpha \subseteq C_{\alpha'}$ for all $0 < \alpha < \alpha' < 1$. This exercise demonstrates an example of UMP tests where this nesting behavior does not occur.

Let $\theta_0, \theta_1 \in \mathbb{R}$ be unequal parameters. Let H_0 denote the hypothesis $\{\theta = \theta_0\}$ and let H_1 denote the hypothesis $\{\theta = \theta_1\}$. Suppose $X \in \{1, 2, 3\}$ is a random variable. If $\theta = \theta_0$, assume that X takes the values $1, 2, 3$ with probabilities $.85, .1, .05$, respectively. If $\theta = \theta_1$, assume that X takes the values $1, 2, 3$ with probabilities $.7, .2, .1$, respectively. Let $\mathcal{T}$ denote the set of hypothesis tests with significance level at most α.

- Let $0 < \alpha < .15$. Show that a UMP class $\mathcal{T}$ test is not unique.
- When $\alpha = .05$, show there is a unique nonrandomized hypothesis UMP class $\mathcal{T}$ test.
- When $\alpha = .1$, show there is a unique nonrandomized hypothesis UMP class $\mathcal{T}$ test.
- Show that the $\alpha = .05$ and $\alpha' = .1$ UMP nonrandomized tests from above do not have nested rejection regions.
- However, when $\alpha = .05$ and $\alpha' = .1$, there are randomized UMP tests $\phi, \phi' \colon \mathbb{R}^n \to [0, 1]$ respectively, that are nested in the sense that $\phi \leq \phi'$.

Solution. We have

$$\frac{f_{\theta_1}(1)}{f_{\theta_0}(1)} = \frac{.7}{.85} = \frac{14}{17}, \qquad \frac{f_{\theta_1}(2)}{f_{\theta_0}(2)} = \frac{.2}{.1} = 2, \qquad \frac{f_{\theta_1}(3)}{f_{\theta_0}(3)} = \frac{.1}{.05} = 2.$$

The Neyman-Pearson Lemma says that likelihood ratio tests $\phi \colon \{1, 2, 3\} \to \mathbb{R}$ of the following form are UMP

$$\phi(x) := \begin{cases} 1 & \text{, if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & \text{, if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ ? & \text{, if } f_{\theta_1}(x) = k f_{\theta_0}(x). \end{cases}$$

So, let us examine those tests for all possible $k > 0$. After examining these different tests, we realize that the case $k = 2$ is of particular interest for this problem, so let us focus on that case.

If $k = 2$, then we have two points $x = 2, 3$ such that we can specify the value of ϕ arbitrarily, while maintaining the UMP property. That is,

$$\phi(x) := \begin{cases} 1 & \text{, if } f_{\theta_1}(x) > k f_{\theta_0}(x) \\ 0 & \text{, if } f_{\theta_1}(x) < k f_{\theta_0}(x) \\ ? & \text{, if } f_{\theta_1}(x) = k f_{\theta_0}(x) \end{cases} = \begin{cases} 0 & \text{, if } x = 1 \\ ? & \text{, if } x = 2, 3. \end{cases}$$

More specifically, for any $0 \leq a, b \leq 1$, $\phi \colon \{1, 2, 3\} \to \mathbb{R}$ is UMP where

$$\phi(x) := \begin{cases} 0 & \text{, if } x = 1 \\ a & \text{, if } x = 2 \\ b & \text{, if } x = 3 \end{cases}$$

A test of this form has power function

$$\beta(\theta) = \mathbf{E}_\theta \phi(X).$$

The significance level of this test is

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0) = .1\phi(2) + .05\phi(3) = (.1)a + (.05)b.$$

If $0 < \alpha < .15$ is our desired significance level, then any choice of $0 \leq a, b \leq 1$ satisfying

$$(.1)a + (.05)b = \alpha$$

is a UMP with significance level α. (The Neyman-Pearson Lemma guarantees this holds.) For a fixed $0 < \alpha < .15$, infinitely many such a, b exist. So, the UMP tests in this case are non-unique.

If $\alpha = .05$, then the set

$$\{(a,b)\colon (.1)a + (.05)b = .05,\ 0 \leq a, b \leq 1\}$$

has a unique element where one of a, b is zero, occurring when $a = 0$ and $b = 1$. So, when $\alpha = .05$, there is a unique nonrandomized UMP test. This test rejects H_0 when $X = 3$.

If $\alpha = .1$, then the set

$$\{(a,b)\colon (.1)a + (.05)b = .1,\ 0 \leq a, b \leq 1\}$$

has a unique element where one of a, b is zero, occurring when $a = 1$ and $b = 0$. So, when $\alpha = .1$, there is a unique nonrandomized UMP test. This test rejects H_0 when $X = 2$.

The above rejection regions are not nested, since the events $\{X = 3\}$ and $\{X = 2\}$ are disjoint.

However, there is are randomized hypothesis tests ϕ, ϕ' with significance level $\alpha = .05, \alpha' = .1$ respectively, such that $\phi \leq \phi'$. For example, we could use

$$\phi(x) := \begin{cases} 0 & \text{, if } x = 1 \\ 1/4 & \text{, if } x = 2 \\ 1/2 & \text{, if } x = 3 \end{cases}, \qquad \phi'(x) := \begin{cases} 0 & \text{, if } x = 1 \\ 1/2 & \text{, if } x = 2 \\ 1 & \text{, if } x = 3 \end{cases}$$

□

Exercise 2.2. Suppose X is a Gaussian distributed random variable with known variance $\sigma^2 > 0$ but unknown mean. Fix $\mu_0, \mu_1 \in \mathbb{R}$. Assume that $\mu_0 - \mu_1 > 0$. We want to test the hypothesis H_0 that $\mu = \mu_0$ versus the hypothesis H_1 that $\mu = \mu_1$. Fix $\alpha \in (0, 1)$. Explicitly describe the UMP test for the class of tests whose significance level is at most α.

Your description of the test should use the function $\Phi(t) := \int_{-\infty}^{t} e^{-x^2/2} dx/\sqrt{2\pi}$, $\Phi\colon \mathbb{R} \to (0, 1)$, and/or the function $\Phi^{-1}\colon (0, 1) \to \mathbb{R}$. (Recall that $\Phi(\Phi^{-1}(s)) = s$ for all $s \in (0, 1)$ and $\Phi^{-1}(\Phi(t)) = t$ for all $t \in \mathbb{R}$.)

Solution. From the Neyman-Pearson Lemma, the UMP is a likelihood ratio test (LRT). Let $k > 0$. (Since $\mathbf{P}_{\theta_0}(f_{\theta_1}(X) = k f_{\theta_0}(X)) = \mathbf{P}_{\theta_1}(f_{\theta_1}(X) = k f_{\theta_0}(X)) = 0$, the UMP is non randomized.) In this case, the LRT has rejection region

$$C := \{x \in \mathbb{R}\colon f_{\theta_1}(x) > k f_{\theta_0}(x)\}.$$

More specifically,

$$
\begin{aligned}
C &:= \{x \in \mathbb{R}\colon f_{\theta_1}(x) > k f_{\theta_0}(x)\} \\
&= \{x \in \mathbb{R}\colon \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma^2}} > k \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu_0)^2}{2\sigma^2}}\} \\
&= \{x \in \mathbb{R}\colon -\frac{(x-\mu_1)^2}{2\sigma^2} > \log(k) - \frac{(x-\mu_0)^2}{2\sigma^2}\} \\
&= \{x \in \mathbb{R}\colon (x-\mu_1)^2 < -2\sigma^2\log(k) + (x-\mu_0)^2\} \\
&= \{x \in \mathbb{R}\colon (x-\mu_1)^2 - (x-\mu_0)^2 < -2\sigma^2\log(k)\} \\
&= \{x \in \mathbb{R}\colon (2x-\mu_1-\mu_0)(\mu_0-\mu_1) < -2\sigma^2\log(k)\} \\
&= \{x \in \mathbb{R}\colon 2x-\mu_1-\mu_0 < -\frac{2\sigma^2\log(k)}{\mu_0-\mu_1}\} \\
&= \{x \in \mathbb{R}\colon x < -\frac{\sigma^2\log(k)}{\mu_0-\mu_1} + \frac{\mu_0+\mu_1}{2}\}.
\end{aligned}
$$

The significance level of this test is

$$
\begin{aligned}
\sup_{\theta\in\Theta_0}\beta(\theta) &= \beta(\mu_0) = \mathbf{P}_{\mu_0}(X \in C) = \mathbf{P}_{\mu_0}(X < -\frac{\sigma^2\log(k)}{\mu_0-\mu_1} + \frac{\mu_0+\mu_1}{2}) \\
&= \mathbf{P}_{\mu_0}(X-\mu_0 < -\frac{\sigma^2\log(k)}{\mu_0-\mu_1} + \frac{\mu_0+\mu_1}{2} - \mu_0) \\
&= \mathbf{P}_{\mu_0}\Big(\frac{X-\mu_0}{\sigma} < -\frac{\sigma\log(k)}{\mu_0-\mu_1} + \frac{\mu_1-\mu_0}{2\sigma}\Big) = \Phi\Big(-\frac{\sigma\log(k)}{\mu_0-\mu_1} + \frac{\mu_1-\mu_0}{2\sigma}\Big)
\end{aligned}
$$

So, if we want a fixed significance level $\alpha \in (0,1)$, then

$$
\Phi^{-1}(\alpha) = -\frac{\sigma\log(k)}{\mu_0-\mu_1} + \frac{\mu_1-\mu_0}{2\sigma}.
$$

That is, we choose k such that

$$
-\Phi^{-1}(\alpha) + \frac{\mu_1-\mu_0}{2\sigma} = \frac{\sigma\log(k)}{\mu_0-\mu_1}.
$$

i.e.

$$
k = \exp\Big(\frac{\mu_1-\mu_0}{\sigma}\Phi^{-1}(\alpha) - \frac{(\mu_0-\mu_1)^2}{2\sigma^2}\Big).
$$

□

Exercise 2.3. This exercise demonstrates that a UMP might not always exists.

Let $X_1, \ldots, X_n$ be i.i.d. Gaussian random variables with known variance and unknown mean $\mu \in \mathbb{R}$. Fix $\mu_0 \in \mathbb{R}$. Let H_0 denote the hypothesis $\{\mu = \mu_0\}$ and let H_1 denote the hypothesis $\mu \neq \mu_0$. Fix $0 < \alpha < 1$. Let $\mathcal{T}$ denote the set of hypothesis tests with significance level at most α. Show that no UMP class $\mathcal{T}$ test exists, using the following strategy.

- Let $\mu_1 < \mu_0$. You may take as given the following fact (that follows from the Karlin-Rubin Theorem): the power at μ_1 is maximized among class $\mathcal{T}$ tests by the hypothesis test ϕ that rejects H_0 when the sample mean satisfies $\overline{X} < c$ for an appropriate choice of $c \in \mathbb{R}$. Assume for the sake of contradiction that a UMP class $\mathcal{T}$ test ϕ' exists.

Then, using the necessity part of the Neyman-Pearson Lemma (i.e. consider testing $\mu = \mu_0$ versus $\mu = \mu_1$), conclude that ϕ' must have the same rejection region as ϕ (just by examining the power of the tests at μ_1.)

- Consider now a test in $\mathcal{T}$ that rejects H_0 when the sample mean satisfies $\overline{X} > c'$ for an appropriate choice of $c' \in \mathbb{R}$. Repeating the previous argument, conclude that ϕ' must reject when $\overline{X} > c'$, leading to a contradiction.

  That is, let $\mu_2 > \mu_0$. You may take as given the following fact (that follows from the Karlin-Rubin Theorem): the power at μ_2 is maximized among class $\mathcal{T}$ tests by the hypothesis test ϕ'' that rejects H_0 when the sample mean satisfies $\overline{X} > c'$ for an appropriate choice of $c' \in \mathbb{R}$. Then, using the necessity part of the Neyman-Pearson Lemma (i.e. consider testing $\mu = \mu_0$ versus $\mu = \mu_2$), conclude that ϕ' must have the same rejection region as ϕ''.

Solution. Since ϕ' is UMP class $\mathcal{T}$ for testing H_0 versus H_1, we have $\beta'(\mu_1) \geq \beta(\mu_1)$. (Here β is the power function of ϕ, and β' is the power function of ϕ'.) From the remark about the Karlin-Rubin Theorem, $\beta'(\mu_1) \leq \beta(\mu_1)$. Therefore, $\beta'(\mu_1) = \beta(\mu_1)$.

Consider $H_1' = \{\mu = \mu_1\}$. Suppose we are testing H_0 versus H_1'. Since $\beta(\mu_1) = \beta'(\mu_1)$, from the Neyman-Pearson Lemma, we must have $\phi' = \phi$ except possibly on a set of probability zero with respect to $\mathbf{P}_{\mu_0}$ and $\mathbf{P}_{\mu_1}$. (Similarly it occurs with probability zero that $f_{\theta_0}(X) = kf_{\theta_1(X)}$ for a constant $k > 0$.) That is, up to probability zero changes to ϕ, both ϕ and ϕ' are nonrandomized hypothesis tests with the same rejection region.

Now, let $\mu_2 > \mu_0$. Since ϕ' is UMP class $\mathcal{T}$ for testing H_0 versus H_1, we have $\beta'(\mu_2) \geq \beta''(\mu_2)$. (Here β' is the power function of ϕ', and β'' is the power function of ϕ''.) From the remark about the Karlin-Rubin Theorem, $\beta'(\mu_2) \leq \beta''(\mu_2)$. Therefore, $\beta''(\mu_2) = \beta'(\mu_2)$.

Consider $H_1'' = \{\mu = \mu_2\}$. Suppose we are testing H_0 versus H_1''. Since $\beta'(\mu_2) = \beta''(\mu_2)$, from the Neyman-Pearson Lemma, we must have $\phi'' = \phi'$ except possibly on a set of probability zero with respect to $\mathbf{P}_{\mu_0}$ and $\mathbf{P}_{\mu_1}$. (Similarly it occurs with probability zero that $f_{\theta_0}(X) = kf_{\theta_1(X)}$ for a constant $k > 0$.) That is, up to probability zero changes to ϕ'', both ϕ'' and ϕ' are nonrandomized hypothesis tests with the same rejection region.

We now have a contradiction, since ϕ' must reject only when $\overline{X} > c$, and ϕ' must reject only when $\overline{X} < c'$.

□

Exercise 2.4. Prove the following version of the Karlin-Rubin Theorem, with the inequalities reversed in the definition of the hypotheses.

Let $\{f_\theta\}$ be a family of PDFs with the MLR property, with respect to a real-valued statistic $Y = t(X)$, where $\theta \in \Theta \subseteq \mathbb{R}$. Let $0 \leq \gamma \leq 1$. Fix $\theta_0 \in \Theta$. Consider the hypothesis $H_0 = \{\theta \geq \theta_0\}$ and the hypothesis $H_1 = \{\theta < \theta_0\}$. Let $c \in \mathbb{R}$. Consider the randomized hypothesis test $\phi\colon \mathbb{R}^n \to [0,1]$ defined by

$$\phi(x) := \begin{cases} 0 & \text{, if } t(x) > c \\ 1 & \text{, if } t(x) < c \\ \gamma & \text{, if } t(x) = c. \end{cases}$$

Define $\alpha := \mathbf{E}_{\theta_0}\phi(X)$. Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most α.

(i) ϕ is UMP class $\mathcal{T}$.

(iii) β, the power function of ϕ, is nonincreasing and strictly decreasing when it takes values in $(0, 1)$.

Proof. We first prove (iii). Let $\theta_1 > \theta_0$ and consider the function $r\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$r(x) := \frac{f_{\theta_1}(x)}{f_{\theta_0}(x)}, \qquad \forall\, x \in \mathbb{R}^n.$$

By assumption, r is a strictly increasing function of $t(x)$. Let $k \in \mathbb{R}$ such that $r(x) = k$ when $t(x) = c$. Since r is a strictly increasing function of $t(x)$, we can rewrite ϕ as

$$\phi(x) = \begin{cases} 0 & , \text{ if } r(x) > k \\ 1 & , \text{ if } r(x) < k \\ \gamma & , \text{ if } r(x) = k. \end{cases}$$

That is, $1-\phi$ is a likelihood ratio test of the hypothesis $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$. Corollary 3.15 from the notes says $1 - \beta(\theta_1) = \mathbf{E}_{\theta_1}(1 - \phi(X)) > 1 - \alpha = \mathbf{E}_{\theta_0}(1 - \phi(X)) = 1 - \beta(\theta_0)$, if $\mathbf{P}_{\theta_0} \neq \mathbf{P}_{\theta_1}$. (If $\mathbf{P}_{\theta_0} = \mathbf{P}_{\theta_1}$, then $\mathbf{E}_{\theta_1}\phi(X) = \mathbf{E}_{\theta_0}\phi(X) \in \{0, 1\}$ since ϕ is either zero or one with probability one in this case, i.e. $\alpha \in \{0, 1\}$.) Assertion (iii) follows.

We now prove (i). First, note that $\alpha = \mathbf{E}_{\theta_0}\phi(X) = \sup_{\theta \geq \theta_0} \mathbf{E}_\theta \phi(X)$ from (iii), so that ϕ is in class $\mathcal{T}$. Now let $\theta_1 < \theta_0$, and let ϕ' be a class $\mathcal{T}$ hypothesis test. By definition of $\mathcal{T}$, $\mathbf{E}_{\theta_0}\phi' \leq \sup_{\theta \geq \theta_0} \mathbf{E}_\theta \phi'(X) \leq \alpha$. So, from the Neyman-Pearson Lemma (sufficiency), ϕ is UMP (in the context of that Lemma), i.e. $\mathbf{E}_{\theta_1}\phi(X) \geq \mathbf{E}_{\theta_1}\phi'(X)$. Since this inequality holds for all $\theta_1 < \theta_0$, we conclude that ϕ is UMP class $\mathcal{T}$, i.e. (i) holds. $\square$

Exercise 2.5. Prove the following one-sided version of the Karlin-Rubin Theorem.

Let $\{f_\theta\}$ be a family of PDFs with the MLR property, with respect to a real-valued statistic $Y = t(X)$, where $\theta \in \Theta \subseteq \mathbb{R}$. Let $0 \leq \gamma \leq 1$. Fix $\theta_0 \in \Theta$. Consider the hypothesis $H_0 = \{\theta = \theta_0\}$ and the hypothesis $H_1 = \{\theta > \theta_0\}$. Let $c \in \mathbb{R}$. Consider the randomized hypothesis test $\phi\colon \mathbb{R}^n \to [0, 1]$ defined by

$$\phi(x) := \begin{cases} 1 & , \text{ if } t(x) > c \\ 0 & , \text{ if } t(x) < c \\ \gamma & , \text{ if } t(x) = c. \end{cases}$$

Define $\alpha := \mathbf{E}_{\theta_0}\phi(X)$. Let $\mathcal{T}$ be the class of all randomized hypothesis tests with significance level at most α.

Then ϕ is UMP class $\mathcal{T}$.

Proof. Let $\theta_1 > \theta_0$. From the Karlin-Rubin Theorem itself (part (iii)), we already know that the power function β of ϕ is nondecreasing. Also, as we proved in the Karlin-Rubin Theorem, if

$$r(x) := \frac{f_{\theta_1}(x)}{f_{\theta_0}(x)}, \qquad \forall\, x \in \mathbb{R}^n,$$

then by assumption, r is a strictly increasing function of $t(x)$. Let $k \in \mathbb{R}$ such that $r(x) = k$ when $t(x) = c$. Since r is a strictly increasing function of $t(x)$, we can rewrite ϕ as

$$\phi(x) = \begin{cases} 1 & , \text{ if } r(x) > k \\ 0 & , \text{ if } r(x) < k \\ \gamma & , \text{ if } r(x) = k. \end{cases}$$

Note that $\alpha = \mathbf{E}_{\theta_0}\phi(X)$ from (iii), so that ϕ is in class $\mathcal{T}$. Let ϕ' be a class $\mathcal{T}$ hypothesis test. By definition of $\mathcal{T}$, $\mathbf{E}_{\theta_0}\phi' \leq \alpha$. So, from the Neyman-Pearson Lemma (sufficiency), ϕ is UMP (in the context of that Lemma), i.e. $\mathbf{E}_{\theta_1}\phi(X) \geq \mathbf{E}_{\theta_1}\phi'(X)$. Since this inequality holds for all $\theta_1 > \theta_0$, we conclude that ϕ is UMP class $\mathcal{T}$. $\square$

Exercise 2.6. Let $X_1, \ldots, X_n$ be i.i.d. random variables. Let $X = (X_1, \ldots, X_n)$. Let $\theta > 0$. Assume that X_1 is uniformly distributed in the interval $[0, \theta]$. Fix $\theta_0 > 0$. Fix $0 < \alpha < 1$. Let $\mathcal{T}$ denote the set of hypothesis tests with significance level at most α.

- Suppose we test $H_0 = \{\theta \leq \theta_0\}$ versus $H_1 = \{\theta > \theta_0\}$. Identify the set of all UMP class $\mathcal{T}$ hypothesis tests.
- Suppose we test $H_0 = \{\theta = \theta_0\}$ versus $H_1 = \{\theta \neq \theta_0\}$. Show there is a unique UMP class $\mathcal{T}$ hypothesis test in this case.

(Hint: first consider testing $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$ with $\theta_1 > \theta_0$, and apply the Neyman-Pearson Lemma. That is, mimic the argument of the Karlin-Rubin Theorem.) (As an aside, observe that, if you naïvely apply the Karlin-Rubin Theorem, you will not find all UMP tests, i.e. a non-strict MLR property version of the Karlin-Rubin Theorem will neglect some UMP tests.)

Solution. The joint distribution of $X_1, \ldots, X_n$ satisfies, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$f_\theta(x) = \prod_{i=1}^{n} \theta^{-1} 1_{[0,\theta]}(x_i) = \theta^{-n} 1_{0 \leq \max_{1 \leq i \leq n} x_i \leq \theta}.$$

Let $\theta_1 > \theta_0$. Then

$$\frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} = \Big(\frac{\theta_1}{\theta_0}\Big)^{-n} \cdot \frac{1_{0 \leq \max_{1 \leq i \leq n} x_i \leq \theta_1}}{1_{0 \leq \max_{1 \leq i \leq n} x_i \leq \theta_0}}.$$

Since $\theta_1 > \theta_0$, evidently this likelihood ratio has the (non-strict) MLR property with respect to $t(x) := \max_{1 \leq i \leq n} x_i$. (As $t(x)$ increases from 0, the ratio of indicator functions is 1, then ∞, then of the form $0/0$, and the latter case is not considered for the MLR property.)

For the moment, suppose we instead test $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$. Then the Neyman-Pearson Lemma says that any (nontrivial) UMP class $\mathcal{T}$ test is a likelihood ratio test of the form

$$\phi(x) := \begin{cases} 1 & \text{, if } \theta_0 < x_{(n)} < \theta_1 \\ \text{arbitrary} & \text{, if } x_{(n)} \leq \theta_0 \text{ or } x_{(n)} \geq \theta_1. \end{cases}$$

(We find these tests by considering different thresholds k in the likelihood ratio tests that reject when $f_{\theta_1}(x) > k f_{\theta_0}(x)$.) The tests of this form that do not depend on θ_1 are of the form

$$\phi(x) := \begin{cases} 1 & \text{, if } \theta_0 < x_{(n)} \\ \text{arbitrary} & \text{, if } \theta_0 \geq x_{(n)}. \end{cases} \qquad (*)$$

Since this test does not depend on θ_1, we conclude that it is UMP for testing $\{\theta = \theta_0\}$ versus $\{\theta > \theta_0\}$ (as in the proof of the Karlin-Rubin Theorem). Again, as in the proof of the Karlin-Rubin Theorem, we conclude that this test is UMP for testing $\{\theta \leq \theta_0\}$ versus $\{\theta > \theta_0\}$. Conversely, any test that is UMP for $\{\theta \leq \theta_0\}$ versus $\{\theta > \theta_0\}$ must be UMP for testing $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$ when $\theta_1 > \theta_0$. Consequently, any UMP for $\{\theta \leq \theta_0\}$ versus $\{\theta > \theta_0\}$ must be of the form $(*)$. The first part of the proof is concluded.

We now prove the second part. If ϕ is UMP for $\{\theta = \theta_0\}$ versus $\{\theta \neq \theta_0\}$, then ϕ must be UMP for testing $\{\theta = \theta_0\}$ versus $\{\theta > \theta_0\}$, as in the Karlin-Rubin Theorem. That is, ϕ must be of the form $(*)$.

Moreover, ϕ must be UMP for testing $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$ with $\theta_1 < \theta_0$. In this case, the Neyman-Pearson Lemma says that any (nontrivial) UMP class $\mathcal{T}$ test is a likelihood ratio test of the form

$$\phi(x) := \begin{cases} 0 & \text{, if } \theta_1 < x_{(n)} < \theta_0 \\ \text{arbitrary} & \text{, if } x_{(n)} \leq \theta_1 \text{ or } x_{(n)} \geq \theta_1. \end{cases} \qquad (**)$$

or

$$\phi(x) := \begin{cases} 1 & \text{, if } x_{(n)} < \theta_1 \\ \text{arbitrary} & \text{, if } x_{(n)} \geq \theta_1. \end{cases} \qquad (***)$$

(We find these tests by considering different thresholds k in the likelihood ratio tests that reject when $f_{\theta_1}(x) > k f_{\theta_0}(x)$. The first type of test occurs when $k = (\theta_0/\theta_1)^n$. The second type of test occurs when $k = 0$.)

If additionally ϕ is of the form $(*)$, (and ϕ does not depend on θ_1) then ϕ must satisfy (for some constant c)

$$\phi(x) := \begin{cases} 1 & \text{, if } \theta_0 < x_{(n)} \text{ or } x_{(n)} < c \\ 0 & \text{otherwise.} \end{cases}$$

(For this particular test ϕ, note that, for any θ_1 satisfying $0 < \theta_1 < \theta_0$, either ϕ is of the form $(**)$ or $(***)$. More specifically, if $c < \theta_1 < \theta_0$, then ϕ is of the form $(**)$ and if $0 < \theta_1 < c$, then ϕ is of the form $(***)$.)

As c changes, so does the significance level α. So, for fixed α, ϕ is unique, as desired.

□

Exercise 2.7. Let $X_1, \ldots, X_n$ be i.i.d. random variables that are uniformly distributed in the interval $[\theta, \theta + 1]$, where $\theta \in \mathbb{R}$ is an unknown parameter. Fix $\theta_0 \in \mathbb{R}$. Suppose we want to test the hypothesis that $\theta \leq \theta_0$ versus $\theta > \theta_0$. For any $0 \leq \alpha \leq 1$, show that there exists a UMP test among tests with significance level at most α, and this test rejects the null hypothesis when $X_{(1)} > \theta_0 + c(\alpha)$ or $X_{(n)} > \theta_0 + 1$.

On the other hand, show that the joint density of $X_1, \ldots, X_n$ does not have the MLR property with respect to any statistic (when $n > 1$). (Hint: if it did have the MLR property, what would the Karlin-Rubin Theorem imply about the UMP rejection regions?)

Solution.

The joint distribution of $X_1, \ldots, X_n$ is

$$f_\theta(x) = \prod_{i=1}^{n} 1_{X_i \in [\theta, \theta+1]} = 1_{X_{(1)}, X_{(n)} \in [\theta, \theta+1]}.$$

Let $\theta_1 > \theta_0$. Then

$$\frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} = \frac{1_{X_{(1)}, X_{(n)} \in [\theta_1, \theta_1+1]}}{1_{X_{(1)}, X_{(n)} \in [\theta_0, \theta_0+1]}}.$$

Observe that this ratio can be $0, 1$ or ∞. More specifically, on the set where at least one of these densities is nonzero, we have

- $f_{\theta_1}(x) > f_{\theta_0}(x)$ when $x_{(n)} > \theta_0 + 1$,

- $f_{\theta_1}(x) = f_{\theta_0}(x)$ when $\theta_1 \leq x_{(1)} \leq x_{(n)} \leq \theta_0 + 1$, and
- $f_{\theta_1}(x) < f_{\theta_0}(x)$ when $x_{(1)} < \theta_1$.

For the moment, suppose we instead test $\{\theta = \theta_0\}$ versus $\{\theta = \theta_1\}$. Then the Neyman-Pearson Lemma says that any (nontrivial) UMP class $\mathcal{T}$ test is a likelihood ratio test of the form

$$\phi(x) := \begin{cases} 1 & \text{, if } x_{(n)} > \theta_0 + 1 \\ \text{arbitrary} & \text{, if } \theta_1 \leq x_{(1)} \leq x_{(n)} \leq \theta_0 + 1 \\ 0 & \text{, if } x_{(1)} < \theta_1. \end{cases}$$

or

$$\phi(x) := \begin{cases} 1 & \text{, if } \theta_1 \leq x_{(1)} \\ \text{arbitrary} & \text{, if } x_{(1)} < \theta_1. \end{cases}$$

The tests of this form that do not depend on θ_1 are of the following form, where $c \in \mathbb{R}$ is a constant:

$$\phi(x) := \begin{cases} 1 & \text{, if } x_{(1)} > \theta_0 + c \ \text{ or } \ x_{(n)} > \theta_0 + 1 \\ 0 & \text{, otherwise.} \end{cases}$$

Since this test does not depend on θ_1, we conclude that it is UMP for testing $\{\theta = \theta_0\}$ versus $\{\theta > \theta_0\}$ (as in the proof of the Karlin-Rubin Theorem). Again, as in the proof of the Karlin-Rubin Theorem, we conclude that this test is UMP for testing $\{\theta \leq \theta_0\}$ versus $\{\theta > \theta_0\}$.

When $n > 1$, the joint density of $X_1, \ldots, X_n$ does not have the MLR property. If it did, then the Karlin-Rubin Theorem would imply that there is a UMP test defined by a single real-valued statistic, but we just showed this is not true.

□

Exercise 2.8. Let $\{f_\theta : \theta \in \mathbb{R}\}$ be a family of positive, single-variable PDFs, i.e. $f_\theta : \mathbb{R} \to (0, \infty)$ for all $\theta \in \mathbb{R}$. Assume that $f_\theta(x)$ is twice continuously differentiable in the parameters θ, x.

Show that $\{f_\theta\}$ has the MLR property with respect to the statistic $t(x) = x$ ($x \in \mathbb{R}$) if and only if

$$\frac{\partial^2}{\partial\theta\partial x} \log f_\theta(x) \geq 0, \qquad \forall\, x, \theta \in \mathbb{R}.$$

Exercise 2.9. Suppose X is a binomial distributed random variable with parameters $n = 100$ and $\theta \in [0, 1]$ where θ is unknown. Suppose we want to test the hypothesis H_0 that $\theta = 1/2$ versus the hypothesis H_1 that $\theta \neq 1/2$. Consider the hypothesis test that rejects the null hypothesis if and only if $|X - 50| > 10$.

Using e.g. the central limit theorem, do the following:

- Give an approximation to the significance level α of this hypothesis test
- Plot an approximation of the power function $\beta(\theta)$ as a function of θ.
- Estimate p values for this test when $X = 50$, and also when $X = 70$ or $X = 90$.

Solution. We have $\alpha = \beta(1/2) = \mathbf{P}_{1/2}(X \in C) = \mathbf{P}_{1/2}(|X - 50| > 10)$. From The Central Limit Theorem, we have the approximation

$$\mathbf{P}_{1/2}(|X - 50| > 10) = \mathbf{P}_{1/2}(\frac{|X - 50|}{(1/2)(10)} > 2) \approx \mathbf{P}(|Z| > 2) \approx .05.$$

Here we used the Matlab command `quad(@(t) (1/sqrt(2*pi))*exp(-t.^2 /2),-2,2)` to get the last probability. So, the significance level of the test is approximately .05. The p-values for this test are roughly

$$p(50) = \mathbf{P}_{1/2}(|X-50| > |50-50|) \approx 1.$$

$$p(70) = \mathbf{P}_{1/2}(|X-50| > |70-50|) = \mathbf{P}_{1/2}(|X-50| > 20) = \mathbf{P}_{1/2}(\frac{|X-50|}{(1/2)(10)} > 4)$$
$$\approx \mathbf{P}(|Z| > 4) \approx 7 \cdot 10^{-5}.$$

Here we used the Matlab command `quad(@(t) (1/sqrt(2*pi))*exp(-t.^2 /2),-4,4)` to get the last probability.

$$p(90) = \mathbf{P}_{1/2}(|X-50| > |90-50|) = \mathbf{P}_{1/2}(|X-50| > 40) = \mathbf{P}_{1/2}(\frac{|X-50|}{(1/2)(10)} > 8)$$
$$\approx \mathbf{P}(|Z| > 8) \approx 5 \cdot 10^{-7}.$$

Here we used the Matlab command `quad(@(t) (1/sqrt(2*pi))*exp(-t.^2 /2),-8,8)` to get the last probability. (I think the actual value of $\mathbf{P}(|Z| > 8)$ is much smaller than this, closer to 10^{-13} though.)

More generally, we have the approximation

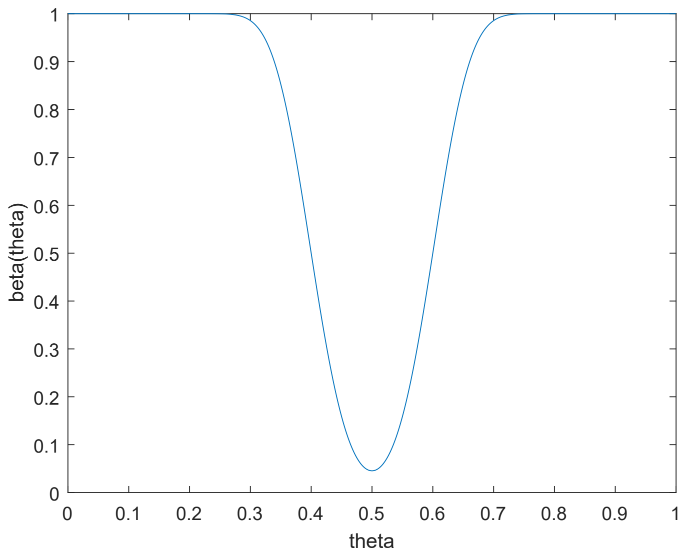
$$\begin{aligned}\beta(\theta) &= \mathbf{P}_\theta(X-50>10) + \mathbf{P}_\theta(X-50<-10)\\ &= \mathbf{P}_\theta\Big(\frac{X-100\theta}{10\sqrt{\theta(1-\theta)}} > \frac{1+5-10\theta}{\sqrt{\theta(1-\theta)}}\Big) + \mathbf{P}_\theta\Big(\frac{X-100\theta}{10\sqrt{\theta(1-\theta)}} < \frac{-1+5-10\theta}{\sqrt{\theta(1-\theta)}}\Big)\\ &\approx \mathbf{P}_\theta\Big(Z > \frac{6-10\theta}{\sqrt{\theta(1-\theta)}}\Big) + \mathbf{P}\Big(Z < \frac{4-10\theta}{\sqrt{\theta(1-\theta)}}\Big)\\ &= 1-\mathbf{P}\Big(Z < \frac{6-10\theta}{\sqrt{\theta(1-\theta)}}\Big) + \mathbf{P}\Big(Z < \frac{4-10\theta}{\sqrt{\theta(1-\theta)}}\Big)\\ &= 1-\Phi\Big(\frac{6-10\theta}{\sqrt{\theta(1-\theta)}}\Big) + \Phi\Big(\frac{4-10\theta}{\sqrt{\theta(1-\theta)}}\Big)\end{aligned}$$

Here we used $\Phi(t) = \mathbf{P}(Z \leq t)$. We can then use following plot in Matlab

```
theta=linspace(0,1,1000);
plot(theta,1-normcdf((6-10*theta)./sqrt(theta.*(1-theta)),0,1)...
+normcdf((4-10*theta)./sqrt(theta.*(1-theta)),0,1));
xlabel('theta');
ylabel('beta(theta)');
```

□

Exercise 2.10. Let $X_1, \ldots, X_n$ be a real-valued random sample of size n from a family of distributions $\{f_\theta : \theta \in \Theta\}$. Suppose $\Theta = \mathbb{R}$. Fix $\theta \in \mathbb{R}$. Denote $X := (X_1, \ldots, X_n)$. Consider a set of hypothesis tests $\phi_\alpha : \mathbb{R}^n \to [0,1]$, for any $\alpha \in [0,1]$. Assume that these tests are nested in the sense that $\phi_\alpha \leq \phi_{\alpha'}$ for all $0 \leq \alpha < \alpha' \leq 1$. Suppose we are testing the hypothesis H_0 that $\{\theta \leq \theta_0\}$ versus H_1 that $\{\theta > \theta_0\}$. Suppose also that $\{f_\theta\}$ has the monotone likelihood ratio property with respect to a statistic $Y = t(X)$ that is a continuous random variable.



- Show that the family of UMP tests with significance level at most α satisfies the nested property mentioned above (for all $\alpha \in [0,1]$).
- Show that, if $X = x$, then the p-value $p(x)$ satisfies

$$p(x) = \mathbf{P}_{\theta_0}(t(X) > t(x)).$$

Solution. The Karlin-Rubin Theorem implies that the UMP tests with significance level at most α are of the form

$$\phi(x) := \begin{cases} 1 & \text{, if } t(x) > c \\ 0 & \text{, if } t(x) < c \\ \gamma & \text{, if } t(x) = c. \end{cases}$$

Since we assume that $Y = t(X)$ is continuous, $t(X) = c$ occurs with probability zero, i.e. we may assume that

$$\phi(x) := \begin{cases} 1 & \text{, if } t(x) > c \\ 0 & \text{, if } t(x) \leq c. \end{cases}$$

The nested property then follows, since as α increases, c decreases.

Denote c_α as the constant $c = c_\alpha$ in the above definition when $\phi = \phi_\alpha$ has significance level α. Recall that significance level α means that

$$\alpha = \sup_{\theta\in\Theta_0} \mathbf{E}_\theta \phi(X) = \sup_{\theta\leq\theta_0} \mathbf{P}_\theta(t(X) > c_\alpha)$$

Since the Karlin-Rubin Theorem implies that the power function is nondecreasing in θ, we have

$$\alpha = \mathbf{P}_{\theta_0}(t(X) > c_\alpha). \qquad (*)$$

We also have

$$p(x) = \inf\{\alpha \in [0,1] \colon \phi_\alpha(x) = 1\} = \inf\{\alpha \in [0,1] \colon t(x) > c_\alpha\}.$$

The nested property implies that $\{\alpha \in [0,1] \colon t(x) > c\}$ is an interval, so that the infimum of this set is the smaller endpoint of that interval. That is, there exists some $\alpha \in [0,1]$ such that $p(x) = \alpha$ and $t(x) = c_\alpha$. So, from $(*)$,

$$\alpha = p(x) = \mathbf{P}_{\theta_0}(t(X) > c_\alpha) = \mathbf{P}_{\theta_0}(t(X) > t(x)).$$

$\square$

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