Graduate Statistics, 545, Spring 2020, USC — Instructor: Steven Heilman

Name: ________________ USC ID: ____________ Date: __________

Signature: ________________.
(By signing here, I certify that I have taken this test while refraining from cheating.)

Mid-Term 2

This exam contains 7 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may *not* use your book or any calculator on this exam. You *cannot* use your homeworks. You are required to show your work on each problem on the exam. The following rules apply:

- You have 50 minutes to complete the exam, starting at the beginning of class.
- **If you use a theorem or proposition from class or the notes or the book you must indicate this** and explain why the theorem may be applied. It is okay to just say, "by some theorem/proposition from class."
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this. Scratch paper appears at the end of the document.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Do not write in the table to the right. Good luck![a]

Reference sheet

Below are some definitions that may be relevant.

Let $\{X_n\}_{n\in\mathbf{Z}}$ be a real-valued stochastic process. We say that $\{X_n\}_{n\in\mathbf{Z}}$ is **weakly stationary** if (i) $\mathbf{E}X_n^2 < \infty$ for all $n \in \mathbf{Z}$, (ii) $\mathbf{E}X_n = \mathbf{E}X_0$ for all $n \in \mathbf{Z}$, and (iii) $\gamma(n, m) = \gamma(n+t, m+t)$ for all $n, m, t \in \mathbf{Z}$.

Let $\{Z_n\}_{n\in\mathbf{Z}}$ be a sequence of random variables. Let $\mu \in \mathbf{R}, \sigma > 0$. We say that $\{Z_n\}_{n\in\mathbf{Z}}$ are **white noise** with mean μ and variance σ^2, denoted $WN(\mu, \sigma^2)$, if $\{Z_n\}_{n\in\mathbf{Z}}$ are a sequence of pairwise uncorrelated, mean μ random variables all with the same variance σ^2.

Let p, q be positive integers. A real-valued stochastic process $\{X_n\}_{n\in\mathbf{Z}}$ is said to be an **ARMA**(p, q) process if $\{X_n\}_{n\in\mathbf{Z}}$ is weakly stationary, and there exist real numbers $a_1, a_2, \ldots, b_1, b_2, \ldots$ and there exists $\{Z_n\}_{n\in\mathbf{Z}}$ that are $WN(0, \sigma^2)$ such that

$$X_n = \sum_{i=1}^{p} a_i X_{n-i} + Z_n + \sum_{j=1}^{q} b_j Z_{n-j}, \qquad \forall\, n \in \mathbf{Z}.$$

If $\mu \in \mathbf{R}$ and if $\{X_n - \mu\}_{n\in\mathbf{Z}}$ is an ARMA(p, q) process, we say that $\{X_n\}_{n\in\mathbf{Z}}$ is an ARMA(p, q) process with mean μ.

For any $z \in \mathbf{C}$, define $\phi(z) := 1 - \sum_{j=1}^{p} a_j z^j$, $\theta(z) := 1 + \sum_{j=1}^{q} b_j z^j$. Let $\{X_n\}_{n\in\mathbf{Z}}$ be a real-valued stochastic process. We define the shift operator S so that $SX_n := SX_{n-1}$, $\forall\, n \in \mathbf{Z}$. Then the definition of ARMA(p, q) process can be rewritten as $\phi(S)X_n = \theta(S)Z_n$, $\forall\, n \in \mathbf{Z}$.

Define the **autocovariance function** $\gamma\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{C}$ by

$$\gamma(n, m) := \mathbf{E}((X_n - \mathbf{E}X_n)\overline{(X_m - \mathbf{E}X_m)}), \qquad \forall\, n, m \in \mathbf{Z}.$$

If $\{X_n\}_{n\in\mathbf{Z}}$ is weakly stationary, this function only depends on one parameter, so we denote its autocovariance function by

$$\gamma(n) := \gamma(n, 0) = \mathbf{E}((X_n - \mathbf{E}X_n)\overline{(X_0 - \mathbf{E}X_0)}), \qquad \forall\, n \in \mathbf{Z}.$$

We say that $\{X_n\}_{n\in\mathbf{Z}}$ is **causal** if there exists a sequence of real numbers $c_0, c_1, \ldots$ with $\sum_{j=0}^{\infty} |c_j| < \infty$ such that $X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}, \qquad \forall\, n \in \mathbf{Z}$.

We say that $\{X_n\}_{n\in\mathbf{Z}}$ is **invertible** if there exists a sequence of real numbers $d_0, d_1, \ldots$ with $\sum_{j=0}^{\infty} |d_j| < \infty$ such that $Z_n = \sum_{j=0}^{\infty} d_j X_{n-j}, \qquad \forall\, n \in \mathbf{Z}$.

1. (10 points) Let $f\colon \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ with $||f||_1 := \int_0^1 |f(x)|\, dx < \infty$. Show that

$$\lim_{n\to\pm\infty} \widehat{f}(n) = 0,$$

(Hint: first, show the assertion holds for any $f \in L_2(\mathbf{R}/\mathbf{Z}, \mathcal{B}, dx)$ as a consequence of Plancherel's Theorem. In particular, the assertion holds for continuous f.)

2. (10 points) Let $N \geq 1$. We define the **Fejér kernel** $F_N \colon \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ to be the function

$$F_N(x) := \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}, \qquad \forall\, x \in \mathbf{R}/\mathbf{Z}.$$

Show that $F_N(x)$ is real valued and nonnegative $\forall\, N \geq 1$, $\forall\, x \in \mathbf{R}/\mathbf{Z}$.

3. (10 points) Let $\{Z_n\}_{n\in\mathbf{Z}}$ be $WN(0,1)$. Consider the MA(1) process $\{X_n\}_{n\in\mathbf{Z}}$ defined by

$$X_n = Z_n - 3Z_{n-1}, \qquad \forall\, n \in \mathbf{Z}.$$

- Show that this process is not invertible (with respect to $\{Z_n\}_{n\in\mathbf{Z}}$).
- Find polynomials $\widetilde{\phi}, \widetilde{\psi}$ and find $\widetilde{\sigma} > 0$, $\{\widetilde{Z}_n\}_{n\in\mathbf{Z}}$ that are $WN(0, \widetilde{\sigma}^2)$ such that

$$\widetilde{\phi}(S)X_n = \widetilde{\theta}(S)\widetilde{Z}_n, \qquad \forall\, n \in \mathbf{Z},$$

  and such that $\{X_n\}_{n\in\mathbf{Z}}$ is invertible (with respect to $\{\widetilde{Z}_n\}_{n\in\mathbf{Z}}$.)

4. (10 points) Let p, q be positive integers. Let $\{X_n\}_{n\in\mathbf{Z}}$ be a real-valued $\mathbf{ARMA}(p, q)$ process. Assume that $\phi(z) \neq 0$ on $\{z \in \mathbf{C}\colon\ |z| = 1\}$. Show that the autocovariance function $\gamma\colon \mathbf{Z} \to \mathbf{R}$ satisfies

$$\sum_{n\in\mathbf{Z}} |\gamma(n)| < \infty.$$

(Scratch paper)