545 Midterm 2 Solutions[1]

1. Question 1

Let $N \geq 1$. We define the **Fejér kernel** $F_N \colon \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ to be the function

$$F_N(x) := \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}, \qquad \forall\, x \in \mathbf{R}/\mathbf{Z}.$$

Show that $F_N(x)$ is real valued and nonnegative $\forall\, N \geq 1$, $\forall\, x \in \mathbf{R}/\mathbf{Z}$.

Solution.

$$F_N = \frac{1}{N}\sum_{\ell=-N}^{N}(N-|\ell|)e_\ell = \frac{1}{N}\sum_{\ell=-N}^{N}\left(\sum_{-N+1\leq j\leq 0\leq k\leq N-1\colon\, j+k=\ell} e_\ell\right)$$
$$= \frac{1}{N}\sum_{\ell=-N}^{N}\left(\sum_{-N+1\leq j\leq 0\leq k\leq N-1\colon\, j+k=\ell} e_j e_k\right) = \frac{1}{N}\sum_{k=0}^{N-1} e_k\,\overline{\Big(\sum_{j=0}^{N-1} e_j\Big)} = \frac{1}{N}\left|\sum_{n=0}^{N-1} e_n\right|^2. \qquad (*)$$

2. Question 2

Let $f \colon \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ with $\|f\|_1 := \int_0^1 |f(x)|\,dx < \infty$. Show that

$$\lim_{n\to\pm\infty} \widehat{f}(n) = 0,$$

Solution. If $\|f\|_2 < \infty$, then Plancherel's Theorem says $\sum_{n\in\mathbf{Z}} |\widehat{f}(n)|^2 < \infty$, so $\lim_{n\to\pm\infty} \widehat{f}(n) = 0$. In particular, if f is continuous, then f is bounded, so $\|f\|_2 < \infty$, hence $\lim_{n\to\pm\infty} \widehat{f}(n) = 0$. Let $\varepsilon > 0$. By e.g. the Weierstrass approximation theorem, let $g \colon \mathbf{R}/\mathbf{Z} \to \mathbf{C}$ be a continuous function such that $\|f-g\|_1 < \varepsilon$. Note that

$$|\widehat{f}(n) - \widehat{g}(n)| = \left|\int_0^1 (f(x)-g(x))e^{-2\pi i n x}dx\right| \leq \|f-g\|_1 < \varepsilon.$$

So,

$$\limsup_{n\to\pm\infty} |\widehat{f}(n)| \leq \limsup_{n\to\pm\infty} |\widehat{g}(n)| + \varepsilon = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\limsup_{n\to\pm\infty} |\widehat{f}(n)| = 0$, as desired.

3. Question 3

Let $\{Z_n\}_{n\in\mathbf{Z}}$ be $WN(0,1)$. Consider the MA(1) process $\{X_n\}_{n\in\mathbf{Z}}$ defined by

$$X_n = Z_n - 3Z_{n-1}, \qquad \forall\, n \in \mathbf{Z}.$$

- Show that this process is not invertible (with respect to $\{Z_n\}_{n\in\mathbf{Z}}$).
- Find polynomials $\widetilde{\phi}, \widetilde{\psi}$ and find $\widetilde{\sigma} > 0$, $\{\widetilde{Z}_n\}_{n\in\mathbf{Z}}$ that are $WN(0,\widetilde{\sigma}^2)$ such that

  $$\widetilde{\phi}(S)X_n = \widetilde{\theta}(S)\widetilde{Z}_n, \qquad \forall\, n \in \mathbf{Z},$$

  and such that $\{X_n\}_{n\in\mathbf{Z}}$ is invertible (with respect to $\{\widetilde{Z}_n\}_{n\in\mathbf{Z}}$.)

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Solution. We have $\theta(1/3) = 0$. So, this process it not invertible by Proposition 7.17 in the notes. (Invertibility means θ must have no zeros in the unit disc.) By repeating the argument of Theorem 7.19 in the notes, we define

$$\widetilde{Z}_n := (1 - S/3)^{-1}(1 - 3S)Z_n = (1 - S/3)^{-1}X_n = \sum_{j=0}^{\infty}(S/3)^j X_n = \sum_{j=0}^{\infty} 3^{-j}X_{n-j}, \qquad \forall\, n \in \mathbf{Z}.$$

It then follows from Theorem 7.19 that $\{X_n\}_{n\in\mathbf{Z}}$ is invertible (with respect to $\{\widetilde{Z}_n\}_{n\in\mathbf{Z}}$. So, the polynomials we use are $\widetilde{\phi}(z) := \phi(z) = 1$ and $\widetilde{\theta}(z) = (1 - z/3)$ for all $z \in \mathbf{C}$.

4. Question 4

Let p, q be positive integers. Let $\{X_n\}_{n\in\mathbf{Z}}$ be a real-valued $\mathbf{ARMA}(p, q)$ process. Assume that $\phi(z) \neq 0$ on $\{z \in \mathbf{C}\colon\, |z| = 1\}$. Show that the autocovariance function $\gamma\colon \mathbf{Z} \to \mathbf{R}$ satisfies

$$\sum_{n\in\mathbf{Z}} |\gamma(n)| < \infty.$$

Solution. Theorem 7.14 in the notes says that there exist constants $\{c_n\}_{n\in\mathbf{Z}}$ with $\sum_{j\in\mathbf{Z}} |c_j| < \infty$ such that $X_n = \sum_{j\in\mathbf{Z}} c_j Z_{n-j}$ for all $n \in \mathbf{Z}$. (The sum converges in L_2 by Proposition 7.5 in the notes.) From Proposition 7.7 in the notes, γ satisfies

$$\gamma(n) = \sum_{j,k\in\mathbf{Z}} c_j c_k \gamma_Z(n - j + k) = \sigma^2 \sum_{j\in\mathbf{Z}} c_j c_{j-n}, \qquad \forall\, n \in \mathbf{Z}.$$

So, $\sum_{n\in\mathbf{Z}} |\gamma(n)| \leq \sigma^2(\sum_{j\in\mathbf{Z}} c_j)^2 < \infty$.